7 Lecture 7: Cosmic Distances

"Science never solves a problem without creating ten more."

George Bernard Shaw

The Big Picture: Last time we introduced the dark energy as the dominant driving mechanism for the cosmic expansion. Today we are going to introduce the redshift as a consequence of expansion of the Universe, and introduce the relevant lengths associated with an expanding Universe.

Redshift

If the wavelength of the emission line in the laboratory is λ_0 and if the observed wavelength is $\lambda > \lambda_0$, then the line is said to be *redshifted* by a fraction z (the *redshift*) given by

$$z = \frac{\lambda - \lambda_0}{\lambda_0}. \tag{160}$$

The redshift is a natural consequence of the Döppler effect — as the Universe expands at a rate a, the wavelength of a particle scales as

$$\lambda = \frac{\lambda_0}{a}, \tag{161}$$

which, combined with eq. (160) yields

$$z = \frac{1-a}{a}, \tag{162}$$

$$a = \frac{1}{1+z}. \tag{163}$$

Gravitational redshift is observed when a receiver is located at a higher gravitational potential than the source. The physical explanation is that the particle loses a fraction of the energy (and hence increases its wavelength) by overcoming the difference in the potential (climbing out of the potential well).

Comoving Coordinates

GR states that the laws of physics are the same in any coordinates. However, some coordinates are easier to work with then others. One such set of coordinates are *comoving coordinates* in which an observer is comoving with the Hubble flow. Only for these observers in the comoving coordinates, the Universe is isotropic (otherwise, portions of the Universe will exhibit a systematic bias: portions of the sky will appear systematically blue- or red-shifted).

Comoving Horizon

Comoving horizon is defined as the total portion of the Universe visible to the observer. It represents the sphere with radius equal to the distance the light could have traveled (in the absence of interactions) since the Big Bang ($t = 0$). In time dt, light travels a comoving distance $d\eta = dx/a = cdt/a$, where dx is a physical distance. After recalling convention adopted earlier $c = 1$, becomes

$$\eta \equiv \int_0^t \frac{dt'}{a(t')}. \tag{164}$$

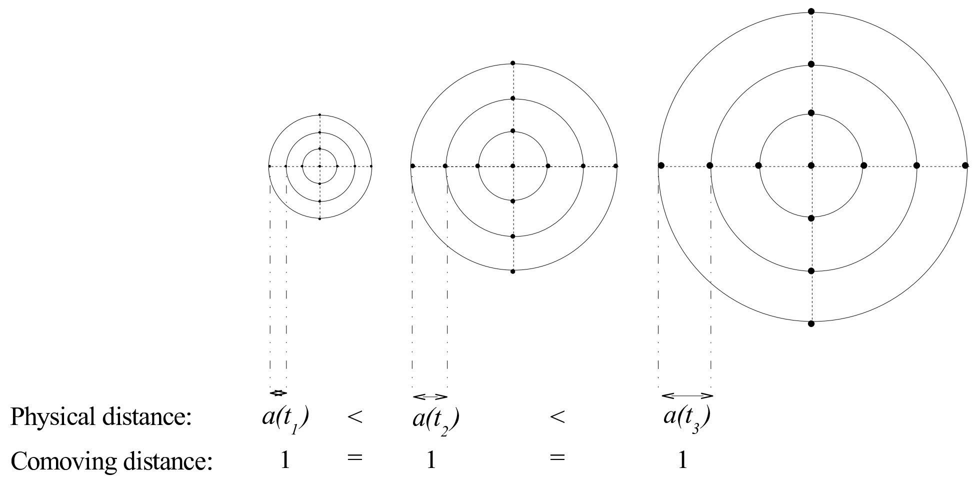

Figure 9: Comoving and physical distances. For an observer located at the center of the circle (stationary in the comoving coordinates), the Universe looks isotropic and homogeneous and it expands in all directions evenly. The comoving coordinates remain fixed, while the physical distance grows as $a(t)$. The two distances are related as $d = ax$, where d is physical and x is comoving distance.

η is called the *conformal time*. Because it is a monotonically increasing variable of time t, it can be used as an independent variable when discussing the evolution of the Universe (just like the time t, temperature T, redshift z and the scale factor a). In some approximations, eq. (164) above can be analytically solved. For instance, in a matter-dominated Universe $\eta \propto a^{1/2}$ and in a radiation-dominated Universe $\eta \propto a$ (Homework set #1).

The importance of the comoving horizon η is in the fact that, under the standard cosmological model, the portions of the sky on our comoving horizon which are separated by more than η are *not causally connected* (there has not been an "exchange of information" between these regions). This means that, in the absence of interaction, these parts should have evolved differently and reached different temperatures. But they are all *very* similar, according to a remarkable isotropy of a few parts in 10^5 in the CMB radiation as measured by the WMAP probe! This is called the *horizon problem.*

The only way to resolve this problem is to allow for all observable matter to have been causally connected early in the history of the Universe.

Inflation

The most obvious way to solve the horizon problem is to allow all matter to interact, and therefore acquire (virtually) the same statistical properties, during the brief period of exponential expansion — *inflation* — immediately following the Big Bang.

Consider an epoch during which the dark energy dominates the matter density: $\Omega_{\rm de} \gg \Omega_{\rm m}$ and $\Omega_{\rm T} \approx \Omega_{\rm de}$, $\Omega_{\rm m} = 0$. If we take $k = 0$, so $\Omega_{\rm T} = \Omega_{\rm de} = 1$, the eq. (154) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2\Omega_{\rm de} = H^2 \qquad \Rightarrow \qquad \dot{a} = Ha \qquad \Rightarrow \qquad a(t) \propto e^{Ht}. \tag{165}$$

This corresponds to a so-called *De Sitter Universe*, characterized by a metric

$$ds^2 = -dt^2 + e^{2Ht}\left[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{166}$$

We are heading toward de Sitter Universe, because the density of dark energy remains constant, while the matter density scales as a^{-3} and radiation density as a^{-4}, which makes the dark energy an ever-increasing part of the cosmic inventory.

The exponential expansion of the scale factor (see eq. (165)) means that the physical distance between any two observers will eventually be growing faster than the speed of light. At that point those two observers will, of course, not be able to have any contact anymore. Eventually, we will not be able to observe any galaxies other than the Milky Way and a handful of others in the gravitationally-bound Local Group cluster of galaxies.

If we consider that the expansion occurred about the time that the strong force "froze out" (at $t = t_{GUT}$), then

$$H \approx \frac{1}{t_{\rm GUT}} \approx \frac{1}{10^{-36}\ {\rm s}} = 10^{36}\ {\rm s}^{-1}, \tag{167}$$

which is an *extremely fast* e-folding time, indicating staggering rate of inflation. In just a few e-folding times, the Universe is already *huge*.

From eq. (150), we have

$$(1 - \Omega_{\rm T}) = -\frac{k}{a^2 H^2}, \tag{168}$$

which means that $\Omega_{\rm T} \to 1$ very fast, regardless of the value of k (recall, we noted earlier that the curvature is relatively unimportant early in the history of the Universe — the behavior of flat, closed and open Universes are asymptotically identical as $t \to 0$). It also means that after inflation $\Omega_{\rm T} = 1$ — the Universe is flat.

We are heading toward de Sitter Universe, because the density of dark energy remains constant, while the energy density of matter drops off as a^3 (see Fig. (8)).

Inflation solves the flatness problem: The WMAP showed that the Universe is flat (or at least *very* nearly flat), *i.e.*, $\Omega_{\rm T} \approx 1$. Why is this so? Why 1? Why not, say, 10^{-5} or 10^6? The standard model does not provide an reasonable explanation for the flat Universe. The problem is exasperated since the $\Omega_{\rm T} = 1$, and thus the flat Universe, is the *unstable fixed point*. This means that if the Universe started with $\Omega_{\rm T} = 1$ *exactly*, it would remain so forever. If, however, the Universe was created with any other value of $\Omega_{\rm T}$, even one arbitrarily close, the separation between the value of $\Omega_{\rm T}$ and 1 would grow over time, presuming only that the scale factor a grows slower then linearly in time. Let us demonstrate this mathematically.

The first Friedmann equation (eq. (101a))

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2, \tag{169}$$

can be rewritten to yield

$$\rho = \frac{3}{8\pi G a^2}\left(\dot{a}^2 + k\right). \tag{170}$$

Dividing by the critical mass

$$\rho_{\rm cr} = \frac{3H^2}{8\pi G}, \tag{171}$$

yields

$$\Omega_{\rm T} = \frac{\rho}{\rho_{\rm cr}} \qquad \Longrightarrow \qquad \Omega_{\rm T} - 1 = \frac{\rho - \rho_{\rm cr}}{\rho_{\rm cr}} = \frac{3k}{8\pi G a^2}\frac{8\pi G a^2}{3\dot{a}^2} = \frac{k}{\dot{a}^2}. \tag{172}$$

It is easily seen that if for $t \to 0$ $\dot{a} \to \infty$ then $\Omega_{\rm T} - 1 \to 0$.

If $a = a_0 \left(\frac{t}{t_0}\right)^p$, then

$$\dot{a} = a_0 t_0^{-p} p t^{p-1}, \tag{173}$$

so that

$$\frac{k}{\dot{a}^2} = \frac{k}{a_0^2 p^2} t_0^{2p} p^2 t^{2(1-p)} \equiv \tilde{k} t^{2(1-p)}. \tag{174}$$

Finally, we obtain

$$\Omega_{\rm T} - 1 = \tilde{k} t^{2(1-p)}, \tag{175}$$

so that $\Omega_{\rm T} - 1 \to 0$ as $t \to 0$ for $p < 1$.

$$\begin{aligned} \Omega_{\rm T} - 1 &\to 0 \quad \text{as} \quad t \to 0 \quad \text{for} \quad p < 1, \\ \Omega_{\rm T} - 1 &\to \infty \quad \text{as} \quad t \to \infty \quad \text{for} \quad p < 1. \end{aligned} \tag{176}$$

This means that the magnitude of $\Omega_{\rm T} - 1$ grows with increasing t. In other words, during the entire history of the Universe over which the scale factor a scales sub-linearly, the Universe is growing increasingly non-flat (unless $\Omega_{\rm T}$ is exactly equal to unity). In the language of mathematics, $\Omega_{\rm T} = 1$ is an *unstable fixed point* for $p < 1$.

Equation (175) holds a clue as to how to naturally obtain a flat Universe, in accordance to observations: change the dynamics so that $\Omega_{\rm T} = 1$ is a *stable fixed point*. All that is required is that the scale factor grows super-linearly (for example $p > 1$ in the equations above). If one allows for a cosmological constant, so that a grows exponentially in time with $a(t) = exp[Ht]$ (eq. (165)), then

$$\dot{a} = He^{Ht}, \tag{177}$$

so that

$$\Omega_{\rm T} - 1 = \frac{\rho - \rho_{\rm cr}}{\rho_{\rm cr}} = \frac{3k}{8\pi G a^2} \frac{8\pi G a^2}{3\dot{a}^2} = \frac{k}{\dot{a}^2} = \frac{k}{H^2} e^{-2Ht}. \tag{178}$$

It follows that any initial deviation from unity is squashed exponentially. If, at some early time in its history, the Universe underwent a period of exponential expansion (inflation), any initial deviation from $\Omega_{\rm T} = 1$ would be reduced to the point *extremely* close to unity, so much so that even the prolonged subsequent evolution with $a \propto t^p$ with $p < 1$, would not drive it appreciably away from it. Therefore, inflation solves the flatness problem.

Distance to an Emitter

It is often useful to determine the distance between a distant emitter and us. In comoving coordinates, the distance to an object at a scale factor a (or alternatively redshift $z = 1/a - 1$) is

$$\chi \equiv \int_{t(a)}^{t(0)} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')}, \tag{179}$$

after the change of variables $da/dt = aH$. For the portion of the Universe which we can observe, which is to about $z \leq 6$, the radiation which dominated early on can be ignored. For the purely matter-dominated flat Universe, we can combine the definition of the Hubble rate $H \equiv \dot{a}/a$ and eq. (115) to obtain

$$H = \frac{\dot{a}}{a} = \frac{2\left(\frac{3H_0}{2}\right)^{2/3} t^{-1/3}}{3\left(\frac{3H_0}{2}\right)^{2/3} t^{2/3}} = \frac{2}{3t} = \frac{2}{3\frac{2}{3H_0} a^{3/2}} = H_0 a^{-3/2}. \tag{180}$$

This simplifies the integral in eq. (179) to

$$\chi^{f,MD}(a) = \frac{1}{H_0}\int_a^1 \frac{da'}{a'^{1/2}}\,\chi = \frac{2}{H_0}a'^{1/2}\Big|_a^1 = \frac{2}{H_0}\left[1 - a^{1/2}\right], \tag{181}$$

where superscripts f and MD denote flat and matter-dominated Universe. In terms of the redshift z eq. (181) becomes (after recalling $z = 1/a - 1$):

$$\chi^{f,MD}(z) = \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]. \tag{182}$$

For small redshift z, $1/\sqrt{1+z} \approx 1 - z/2$, so $\chi^f(z) \approx z/H_0$. For large redshift z, $\chi(z) \to 2/H_0$.

Angular Diameter Distance

Another important distance in astronomy is the *angular diameter distance*. In astronomy, the angular diameter distance is determined by measuring the angle θ subtended by an object of known physical size l. Assuming that the angle is small, it is given by

$$d_A = \frac{l}{\theta}. \tag{183}$$

To compute the angular diameter distance in an expanding Universe, we express the quantities l and θ in comoving coordinates. The comoving size of an object of physical size l is simply l/a, while the angle subtended in the flat Friedmann Universe is

$$\theta = \frac{\frac{l}{a}}{\chi(a)}, \tag{184}$$

so finally we have

$$d_A^{f,MD} = a\chi = \frac{\chi}{1+z}. \tag{185}$$

For small redshift z, $d_A^{f,MD} \approx \chi$. At large z, $d_A^{f,MD} \to \chi/z \to 2/(zH_0)$, so the angular diameter distance decreases with redshift z. This means that the in the flat Universe, objects at large redshifts appear larger than they would at intermediate redshifts!

Luminosity Distance

In astronomy, distances can be inferred by measuring the flux from an object of known luminosity ("standard candles"). Flux and luminosity are related through

$$F \equiv \frac{L}{4\pi d^2}, \tag{186}$$

since the total luminosity through a spherical constant with area $4\pi d^2$ is constant. The total luminosity is defined as the amount of energy radiated per unit time. This means $L \equiv \frac{dE}{dt}$. Assuming that, without loss of generality, all the N photons radiated have the same frequency ν (wavelength λ). Then the luminosity becomes $L = \frac{\hbar}{\lambda}\frac{dN}{dt}$. In comoving coordinates $\lambda_c = \lambda/a$ and the t-derivative is replaced by η-derivative (recall $dt = a d\eta$), so

$$L(\chi) = \frac{\hbar}{\lambda_c}\frac{dN}{d\eta} = a\frac{\hbar}{\lambda}\frac{dN}{dt}a = \frac{\hbar}{\lambda}\frac{dN}{dt}a^2 = La^2. \tag{187}$$

Then the observed flux is

$$F = \frac{La^2}{4\pi\chi^2} = \frac{L}{4\pi\left(\frac{\chi}{a}\right)^2} \equiv \frac{L}{4\pi d_L^2}, \tag{188}$$

where

$$d_L \equiv \frac{\chi}{a}, \tag{189}$$

is the *luminosity distance*.

All three distances discussed today — conformal, angular diameter and luminosity — are larger in a Universe with a cosmological constant than in the one without. You will convince yourself (and me, I hope) of this in one of the problems from your Homework set #1.

Important note: reliable measurements of these distances, when combined with accurate measurements of the redshift z can provide a constraint on the energy density of the dark energy $\Omega_{\rm de0}$ (as will be discussed later in more detail).

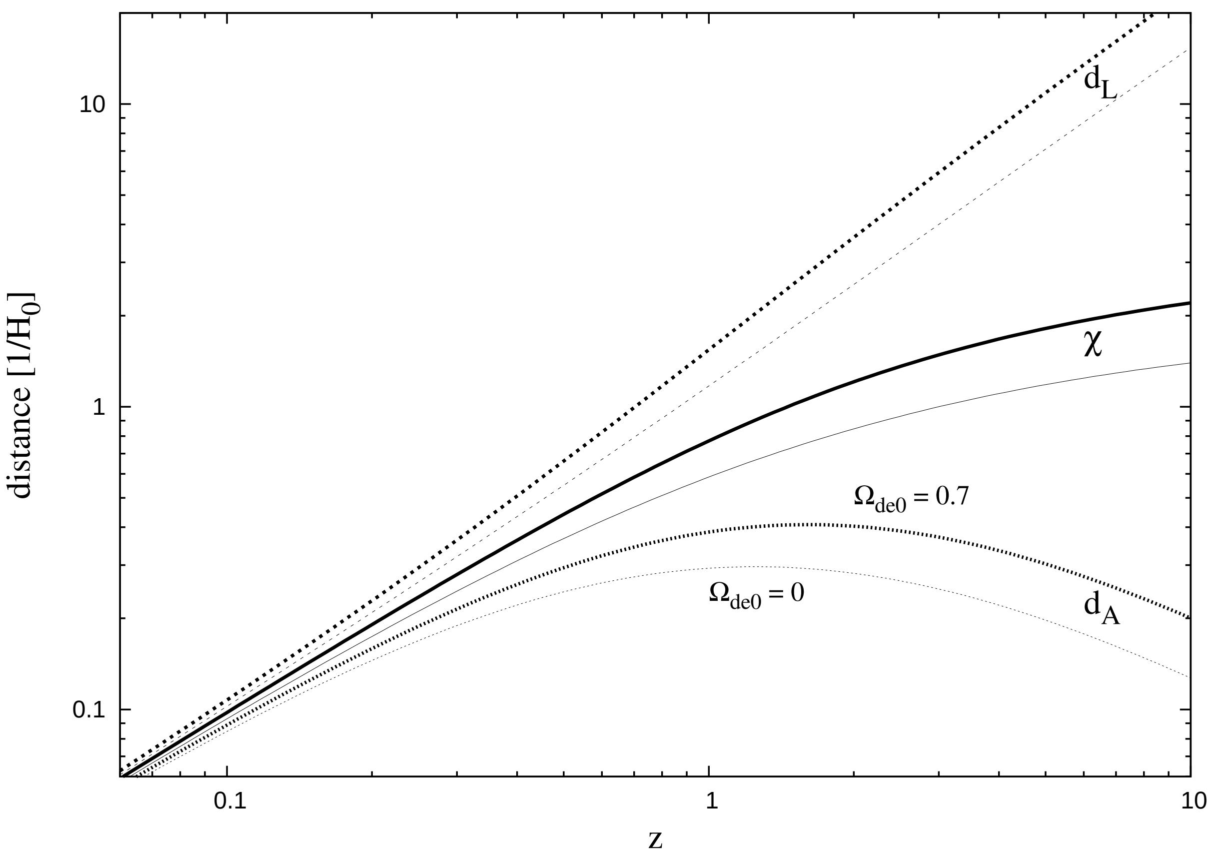

Figure 10: Three distances measures in a flat expanding matter-dominated Universe (thin lines) and Universe with matter and dark energy corresponding to $\Omega_{\rm de0} = 0.7$ (thick lines). Solid lines correspond to the comoving distance χ, dotted lines to angular diameter distance d_A, and dashed lines to luminosity distance d_L.