Chaotic Mixing in Charged-Particle Beams and Galaxies

Courtlandt L. Bohn*
Fermilab, Batavia, IL 60510
Henry E. Kandrup
University of Florida, Gainesville, FL 32611
Rami A. Kishek
University of Maryland, College Park, MD 20742

Self-interacting, nonequilibrium, very-many-body systems such as elliptical galaxies and charged-particle beams seem generically to exhibit rapid evolution to a quasi-equilibrium state. Such systems comprise some 10^{10-12} particles. The associated collisional relaxation time of elliptical galaxies is $\sim 10^{15-16}$ years, several orders of magnitude larger than the age of the universe. For a nonrelativistic charged-particle beam it is $\sim 1-10\ \mu$s, i.e., "1-10 km," typically much longer than, e.g., the length of a linac. Yet, elliptical galaxies appear to be "relaxed" to a smooth density distribution, and charged-particle beams have likewise been seen to "relax" to a smooth density distribution, and also to equipartition in a few meters, depending on details of the space charge. How so? This paper focuses on the behavior of the orbits comprising the system and how these orbits mix through their accessible phase space. Time scales for relaxation to quasi-equilibria are estimated; the estimates are in reasonable agreement with the true values computed in numerical simulations and seen in (the few) beam experiments done to date.

1. Introduction

Rapid irreversible dynamics is a practical concern in producing high-brightness charged-particle beams. Time scales of irreversible processes place constraints on methods for compensating against degradation of beam quality caused by, for example, space charge or coherent synchrotron radiation. Compensation must be fast compared to active irreversible processes, and this affects the choice and configuration of the associated hardware.

A beam bunch with space charge comprises an N-body system with typically $3N$ degrees of freedom. Orbits in the nonlinear space-charge force may be chaotic, especially in a nonequilibrium beam. Through phase mixing, an initially localized ensemble of chaotic orbits will grow exponentially and eventually diffuse through its accessible phase space, reaching an invariant distribution. This is what is meant by "chaotic mixing" [1]. The process is irreversible in the sense that infinitesimal fine-tuning is needed to reassemble the initial conditions. It is also distinctly different from phase mixing of regular orbits, a process that winds an initially localized ensemble into a filament over a comparatively narrow region of phase space, and that is in principle reversible. Whereas chaotic mixing proceeds exponentially over a well-defined time scale and causes global, macroscopic changes in the system, phase mixing carries an algebraic time dependence, proceeds on a time scale depending on the distribution of orbital frequencies across the ensemble, and acts only over a portion of the phase space.

Chaotic mixing may or may not be rapid. For example, simulations of large self-gravitating N-body systems in which the smoothed density is constant over a stationary ellipsoidal volume show that the orbits, though they are chaotic, behave for very long times as if they were regular [2]. These simulations, however, also reveal that adding a density cusp and/or inserting a massive black hole at the centroid can greatly accelerate chaotic mixing, driving it to completion within a few orbital periods. The process tends to make the distribution of stars more isotropic [3], reminiscent of equipartitioning in beams. In short, structure in the density distribution of a self-gravitating system can lead to rapid chaotic mixing by increasing the degree of chaoticity of the orbits.

*clbohn@fnal.gov

By analogy, one might conjecture that structure in the density distribution of a self-interacting beam can likewise lead to rapid chaotic mixing. One example is the University of Maryland five-beamlet experiment that showed presumably irreversible dissipation of the beamlets after a few space-charge-depressed betatron periods [4]. Simulations of the experiments revealed a substantial fraction of globally chaotic orbits [5], and chaotic mixing thereby presents itself as a possible mechanism. In any case, ascertaining conditions that lead to rapid chaotic mixing in beams is an undertaking of practical importance.

2. Theory vs. Numerical Experiments

The past few years have seen development of a geometric method proposed by M. Pettini to quantify chaotic instability in Hamiltonian systems with many degrees of freedom. The central idea is to describe the dynamics in terms of average curvature properties of the manifold in which the particle orbits are geodesics. The method hinges on the following assumptions and approximations, which are discussed thoroughly in [6]: (1) a generic geodesic is chaotic; (2) the manifold's effective curvature is locally deformed but otherwise constant; (3) the effective curvature reflects a gaussian stochastic process; and (4) long-time-averaged properties of the curvature are calculable as phase–space averages over an invariant measure, specifically, the microcanonical ensemble. The gaussian process is the zeroth-order term in a cumulant expansion of the actual stochastic process; assumption (3) is that the zeroth-order term suffices. The end result relates chaotic instability to the geometric properties of the manifold defined by the long-time-averaged orbits. Though the assumptions and approximations lack universal validity and are difficult to prove rigorously for a given system, they nonetheless would seem to offer a reasonable basis for identifying conditions that can produce rapid chaotic mixing [7].

With the assumptions and approximations, Pettini and others [6] derive an expression for the largest Lyapunov exponent χ (which is a measure of the mixing rate [7]) in terms of the curvature and its standard deviation averaged over the microcanonical ensemble. The idea is that, as $t \to \infty$, chaotic orbits of total energy E mix through the configuration space toward an invariant measure, taken per assumption (4) to be the microcanonical ensemble $\delta(H-E)$, over which time averages become equivalent to phase-space averages. Specifically, for an arbitrary function $A(\mathbf{q})$, the averaging process is

$$\langle A \rangle \equiv \lim_{t\to\infty} \langle A \rangle_t = \frac{\int d\mathbf{q} \int d\dot{\mathbf{q}} A(\mathbf{q}) \delta[H(\mathbf{q},\dot{\mathbf{q}}) - E]}{\int d\mathbf{q} \int d\dot{\mathbf{q}}\, \delta[H(\mathbf{q},\dot{\mathbf{q}}) - E]}. \tag{1}$$

Pettini *et. al.*'s method yields

$$\chi(\rho) = \frac{1}{\sqrt{3}} \frac{L^2(\rho) - 1}{L(\rho)} \sqrt{\kappa} \quad L(\rho) = \left[T(\rho) + \sqrt{1 + T^2(\rho)} \right]^{1/3}, \quad T(\rho) = \frac{3\pi\sqrt{3}}{8} \frac{\rho^2}{2\sqrt{1+\rho} + \pi\rho}, \tag{2}$$

in which $\rho \equiv \sigma/\kappa$, a quantity that measures the ratio of the average curvature radius to the length scale of fluctuations, with

$$\kappa = \frac{\langle \Delta V \rangle}{3N-1}, \quad \sigma = \sqrt{\frac{\langle (\Delta V)^2 - \langle \Delta V \rangle^2 \rangle}{3N-1}}, \tag{3}$$

in which Δ denotes the Laplacian $\partial_i \partial^i$.

The geometric quantities derive from the $6N$-dimensional microcanonical ensemble. Anticipating that granularity takes a long time to affect mixing, and wishing to identify conditions for rapid mixing, we now consider the influence of the 3-dimensional coarse-grained space-charge potential V_s on a generic chaotic orbit. We presume the assumptions and approximations carry over to the coarse-grained system; when they do not, chaotic mixing will normally be too slow to be of concern. We take the external focusing potential V_f to be quadratic in the coordinates $\mathbf{x}$ comoving with the bunch, *i.e.*, $V_f(\mathbf{x}) = (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$; the total potential is $V = V_f + V_s$. Per Equation (3) and Poisson's equation the quantities κ and σ are determined from $\nabla^2 V = \omega_f^2 - \omega_p^2(\mathbf{x})$, in which $\omega_f^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$, $\omega_p^2(\mathbf{x}) = n(\mathbf{x})e^2/(\varepsilon_o m)$, $n(\mathbf{x})$ is the (smoothed) particle density, e and m are the single-particle charge and mass, respectively, and ε_o is the permittivity of free space. With $\omega_{p0} \equiv \omega_p(0)$, the results may be expressed conveniently in terms of the space-charge-depressed focusing strength $\omega_s^2 = \omega_f^2 - \omega_{p0}^2$ and the normalized particle density $\nu(\mathbf{x}) = n(\mathbf{x})/n(0)$ as $\kappa = (\omega_{p0}^2/2)\left[(\omega_s/\omega_{p0})^2 + 1 - \langle\nu\rangle\right]$, $\sigma = \omega_{p0}^2\sqrt{(\langle\nu^2\rangle - \langle\nu\rangle^2)/2}$. Inserting these results into Equation (2) gives the associated time scale, $t_m \equiv 1/\chi$, for irreversible chaotic mixing. When the standard deviation of the density distribution is large, as can be the case when substructure is present, ρ will be appreciable, and in turn Equation (2) makes clear that t_m will be a few space-charge-depressed betatron periods. This is consistent with, *e.g.*, the aforementioned University of Maryland experiment showing irreversible dissolution of both matched and mismatched 5-beamlet configurations over a few depressed betatron periods [4].

The aforementioned studies of galactic dynamics permit a more precise means of assessing the theory. Comprehensive simulations of chaotic mixing in galaxies consisting of a homogeneous ellipsoid with a massive black hole at its centroid have recently been done [8]. Comparison of these results against those of the theory reveal that the analytic results agree closely with the numerical results, particularly for intermediate–to–small values of the black-hole mass [7]. The agreement suggests that the 6-dimensional phase space governed by the potential exhibits global chaos and associated rapid irreversible mixing over the bulk of the parameter space. Uncertainty in the calculated time scale seems to be principally associated with uncertainty in the autocorrelation time; it is comparatively insensitive to the choice of the invariant measure that weights the statistical averages.

Preliminary results from a numerical study in progress indicate that chaotic mixing is associated with the equipartitioning of anisotropic charged-particle beams [9]. The study is based on the same methodology as that of the galactic studies, *viz.*, following the evolution of initially localized ensembles, looking for exponential divergence of orbits in the phase space, and deciphering the time scale for the divergence. The results suggest that anisotropy establishes a significant population of chaotic orbits, these orbits diverge exponentially, and the divergence saturates on a global scale as the orbits fill their accessible phase space.

3. Synopsis

To summarize, investigations to date point to the presence of chaotic orbits in nonequilibrium systems comprising a large number of mutually interacting particles. The chaotic behavior arises generically from a parametric instability that can be modeled by a stochastic-oscillator equation. Calculated time scales are normally reasonably close to those seen in numerical experiments and are consistent with (the few) existing laboratory experiments concerning charged-particle beams. However, the theoretical treatment provides no information as to what criteria are necessary and sufficient to establish a preponderance of globally chaotic orbits; it merely hypothesizes their existence. Likewise, it fails to account for "sticky" chaotic-orbit segments that, when present, tend to slow the mixing. Real systems may, however, mitigate this *caveat*. For example, external noise is known to add greatly to the efficiency of chaotic mixing by overcoming stickiness. Localized irregularities that have been coarse-grained away may likewise increase the chaoticity of the orbits. The lower limit corresponds to graininess manifesting itself in binary particle interactions that, in both regular and chaotic smoothed potentials, appears to constitute a source of noise. Graininess establishes diffusion of an orbit from the trajectory it would have in the smooth potential. The diffusion proceeds as a power law in time for regular orbits, but exponentially for chaotic orbits [10].

When chaotic mixing is active, structure in the density distribution determines how rapidly it progresses. Production of high-brightness beams may lead to transient, localized density peaks, as has been seen, *e.g.*, during bunch compression and in merging multiple beamlets. Thus, an accelerator designer who cannot know *a priori* the detailed bunch structure will want to ensure that emittance compensation is completed within roughly a plasma period to be confident that irreversible mixing will not spoil the compensation. This criterion translates into permissible beamline locations and maximum lengths that the associated hardware can occupy [11].

An interesting possibility is to design laboratory experiments involving beams with an eye toward applications to other areas, such as galaxies for which direct experimentation is obviously impossible, or large N-body systems of interacting particles in general. We are in the process of designing such experiments to be conducted with the University of Maryland Electron Ring.

Acknowledgments

This research was supported by NSF grant AST-0070809, and by DOE grants DE-FG02-94ER40855, DE-FG02-92ER54178, and DE-AC02-76CH00300, the latter being through the Universities Research Association, Inc.

References

[1] D. Merritt and M. Valluri, Astrophys. J. **471**, 82 (1996).
[2] M. Valluri and D. Merritt, in *The Chaotic Universe*, eds. V.G. Gurzadyan and R. Ruffini, pp. 229–244 (World Scientific, Singapore, 2000).
[3] D. Merritt and G.D. Quinlan, Astrophys. J. **498**, 625 (1998).
[4] I. Haber, D. Kehne, M. Reiser, and H. Rudd, Phys. Rev. A **44**, 5194 (1991); M. Reiser, *Theory and Design of Charged Particle Beams*, (John Wiley & Sons, New York, 1994), §6.2.2.
[5] D. Kehne (private communication).
[6] M. Pettini, Phys. Rev. E **47**, 828 (1993); L. Casetti, C. Clementi, and M. Pettini, Phys. Rev. E **54**, 5969 (1996); P. Cipriani and M. Di Bari, Planet. Space Sci. **46**, 1499 (1998); L. Casetti, M. Pettini, and E.G.D. Cohen, Phys. Rep. **337**, 237 (2000).
[7] The theoretical foundations are explored in H.E. Kandrup, I.V. Sideris, and C.L. Bohn, Phys. Rev. E (submitted).
[8] H.E. Kandrup and I.V. Sideris, Celestial Mechanics and Dynamical Astronomy (in press).
[9] R.A. Kishek, *et al.* Proc. 2001 Part. Accel. Conf. (2001).
[10] H.E. Kandrup and I.V. Sideris, Phys. Rev. E (in press), and follow-on work.
[11] C. L. Bohn, in *The Physics of High Brightness Beams*, edited by J. Rosenzweig and L. Serafini (World Scientific, Singapore, 2000), pp. 358–368.