PHASE MIXING OF CHAOTIC ORBITS AS AN IRREVERSIBLE "RELAXATION" MECHANISM*

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Abstract

Orbits that are chaotic will tend to phase-mix exponentially through their accessible phase space. This phenomenon, commonly called "chaotic mixing", stands in marked contrast to phase mixing of regular orbits. It is inherently irreversible, and thus its associated e-folding time scale sets a condition on any process envisioned for emittance compensation. Accordingly, two questions arise. First, under what conditions does chaotic mixing manifest itself in beams? Second, when it is active, over what time scale does it operate? The work described here is part of an ongoing effort to answer these questions.

1 INTRODUCTION

We adopt the viewpoint that, under the influence of space charge, the evolution of beams, and of confined nonneutral plasmas in general, may be understood in terms of phase mixing of the constituent particle orbits. For example, linear Landau damping is merely phase mixing of regular orbits [1], a process by which initially neighboring orbits diverge secularly, i.e., as a power law in time [2]. A given space-charge potential may or may not support a population of globally chaotic orbits, i.e., orbits that wander over a large portion of their accessible phase space. Initially neighboring globally chaotic orbits fill their accessible phase space exponentially, a process known as "chaotic mixing" that was initially conceived in the astrophysical context of galactic dynamics [3,4]. When a substantial population of globally chaotic orbits exists, it irreversibly dissipates correlations. In beams the consequence is an irreversible emittance growth [5].

Inasmuch as chaotic mixing is irreversible and acts exponentially, it is essential to identify conditions for its presence in beams, and to quantify the time scale of the associated dynamics. In connection with coherent synchrotron radiation, the subject of this Workshop, one needs to bracket the conditions for compensating against the correlated emittance growth it can generate. This paper constitutes a status report on a collaborative investigation oriented toward deciphering conditions that support chaotic orbits, and time scales for chaotic mixing.

2 RESULTS OF THEORY

A semianalytic theory exists that relies on assumptions of ergodicity and a microcanonical distribution to estimate the largest Lyapunov exponents, i.e., the chaotic-mixing rates, in lower-dimensional, e.g., fully coarse-grained, time-independent Hamiltonian systems [5]. Chaos arises generically from a parametric instability that can be modeled by a stochastic-oscillator equation; linearized perturbations of a chaotic orbit satisfy a harmonic-oscillator equation with a randomly varying frequency. The underlying assumptions are, strictly speaking, invalid, yet the theory commonly yields estimates that are good to within a factor ~2 [6].

Applied to space-charge potentials, the theory yields an estimate of the chaotic-mixing rate λ as:

$$\lambda(\rho) \simeq (\kappa/3)^{1/2}\,[L^2(\rho) - 1]/L(\rho); \qquad (1)$$

in which

$$\kappa = (\omega_f^2 - \omega_{po}^2\langle\nu\rangle)/2,$$
$$L(\rho) = \{T(\rho) + [1+T^2(\rho)]^{1/2}\}^{1/3},$$
$$T(\rho) = (3^{3/2}\pi\rho^2)/\{8[2(1+\rho)^{1/2} + \pi\rho]\},$$
$$\rho = [2(\langle\nu^2\rangle - \langle\nu\rangle^2)]^{1/2}/[(\omega_f/\omega_{po})^2 - \langle\nu\rangle];$$

$\omega_f = (\omega_x^2 + \omega_y^2 + \omega_z^2)^{1/2}$ and ω_{po} refer to the external focusing frequency and the plasma frequency measured at the system's centroid, respectively, ν is the density normalized to the centroid density, and "$\langle q\rangle$" denotes a phase-space average of quantity q weighted by the microcanonical ensemble. In a system that is moderately out of equilibrium, one would expect to have $\rho \sim 1$ typically, for which $\lambda/f \sim 0.82$, with $f \equiv \kappa^{1/2}/(2\pi)$ representing the "dynamical frequency", i.e., the average orbital frequency. Thus, in such systems, the chaotic-mixing time scale is roughly one dynamical time.

This result, combined with concerns about regular phase mixing of large correlations as well as the impracticality of knowing the detailed phase-space structure of a beam at every point along an accelerator, leads to a conservative criterion for successful emittance compensation. Specifically, for one to be reasonably sure of its efficacy, a process of emittance compensation, i.e., removal of correlations within the beam, should be completed within a plasma period as measured from the source of the correlations. Expressed in terms of beam parameters, this criterion is [5]

$$K(\text{MeV}) > 2.5\{Q(\text{nC})[H(\text{m})]^2/[X(\text{mm})Y(\text{mm})Z(\text{mm})]\}^{1/3},$$

in which K is the beam's kinetic energy, Q is the bunch charge, H is the beamline length occupied by the emittance-compensation hardware, and (X,Y,Z) are the root-mean-square (x,y,z)-dimensions of the bunch, respectively. Let us hypothesize that the hardware occupies H = 5 m. Then, for example, given beam parameters of Jefferson Lab's IRFEL Demo [7] (Q ~ 0.1

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nC, rms beam dimensions ~ 1 mm), compensation should succeed at T > 3 MeV, i.e., anywhere after the injector. And for beam parameters of a TESLA FEL (Q ~ 1 nC, rms beam dimensions ~ 0.1 mm), compensation should succeed at T > 70 MeV. The injector envisioned for the TESLA FEL will deliver ~140 MeV beam [8], so emittance compensation downstream from the injector would appear to be viable.

3 NUMERICAL EXPERIMENTS

3.1 Equipartitioning

In a recent computational study using the 2-1/2 D version of the particle-in-cell code WARP, we discovered strong evidence that chaotic mixing is intimately connected with equipartitioning in beams [9]. This work concerned a highly space-charge-dominated, direct-current, cylindrical beam in which the initial momentum space reflected an anisotropic pressure such that $p_{xx}=2p_{yy}$. As the beam evolved, the pressure became increasingly isotropic on a rapid time scale. Though the relaxation time for two-body collisions in this beam corresponds to a propagation distance ~1 km, the beam equipartitioned in only ~5 m, followed by anisotropic pressure oscillations that largely damped by ~50 m. The underlying dynamics is manifestly collision*less*. The equipartitioning time scales were seen to correlate with the evolution of initially localized ensembles of particles. As indicated in Fig. 1 on the next page, these ensembles expanded exponentially with an e-folding "time" ~2 m, which is about two plasma periods, and filled their accessible phase spaces in ~50 m. Moreover, plots of individual orbits appeared to reflect globally chaotic behaviour in keeping with the exponential dynamics. The beam parameters for this experiment were a plasma period of 1.14 m and x and y betatron focusing periods without space charge of 1.63 m. Inserting these numbers, along with ρ=1 and $\langle v\rangle$=1, into Eq. (1) yields an estimated e-folding (mixing) time ~2.0 m, in fortuitous agreement with the simulation.

This first study comprises a form of "symmetry breaking", wherein the broken symmetry is in momentum space rather than configuration space. The beam thus begins in a nonequilibrium state, and it evolves toward a metaequilibrium in which the particle orbits have filled an invariant measure of phase space. The transient dynamics reflects an intricate, evolving network of space-charge waves that set up a complicated potential in which a substantial population of particle orbits becomes globally chaotic. By contrast, the symmetric, isotropic system establishes a potential that is integrable, in which the orbits are accordingly regular.

3.2 Five Beamlets in Smooth Transport Channel

A well-known experiment in accelerator physics is that of M. Reiser and collaborators [10] concerning the propagation of five beamlets in a periodic solenoidal transport channel. The beam is nonrelativistic and subject to considerable space-charge forces. The relaxation time via two-body collisions in this beam corresponds to a propagation distance ~1 km. Yet, regardless how well the beam was root-mean-square (rms) matched to the transport channel, the beamlets were seen to reappear only once, at a point ~1 m from the source. Their failure to reappear again would seem to reflect a collisionless process that, in effect, causes the particle orbits to lose memory of their initial conditions. As discussed in Ref. [10], simulations with a particle-in-cell code well reproduced the measurements.

To explore how chaotic mixing influences the dynamics of such a manifestly nonequilibrium beam, we simulated the experiment using WARP. Our simulation differed from the experiment only in that we took the transport channel to impart a constant, linear external focusing force, whereas in the experiment the channel comprised a periodic solenoidal focusing lattice. Nonetheless, our simulation results correlate well with the measurements.

The strongly time-dependent space-charge potential drives a large population of globally chaotic orbits. Figure 2 on the next page illustrates how orbits of representative test particles evolve. The test particles interact with the potential but not with each other. One sees that typical ensembles that are initially localized in phase space grow exponentially to fill much of their respective accessible regions of phase space. Meanwhile the five beamlets lose their identity.

In this experiment, though chaotic orbits are easily found, it is difficult to separate the macroscopic influence of chaotic mixing from that of linear phase mixing of the five beamlets. Because the beamlets are large, they span a broad band of orbital frequencies in the initial potential. Accordingly, they smear through large regions of phase space and quickly overlap. One can be sure, however, that chaotic mixing is active over the bulk of phase space.

Analogous behaviour is seen in simulations of a rms-mismatched five-beamlet system, except now there is an additional phenomenon, namely, the formation of a prominent halo. Indications from the simulation are that the halo forms via parametric resonance with oscillations of the global potential as envisioned by Gluckstern [11]. Yet microscopic processes can stochastically convert core orbits to halo orbits and vice versa, thereby providing a mechanism for the production of "new halo" [12].

3.3 Chaos in Time-Independent Potentials

In the spirit of trying to decipher conditions that lead to chaos in beams, we now explore time-independent potentials in thermal equilibrium [13]. The corresponding density profiles for beams in which space charge is strong are constant near the bunch centroid, and at larger radii they decay over a Debye length to a low-density tail. In the Debye fall-off, the net force on a particle is nonlinear, and it is of interest to determine whether this force can support a substantial population of chaotic orbits. Of course, all spherically symmetric systems are integrable and support only regular orbits. In general, however, a system will be aspherical because the external focusing is

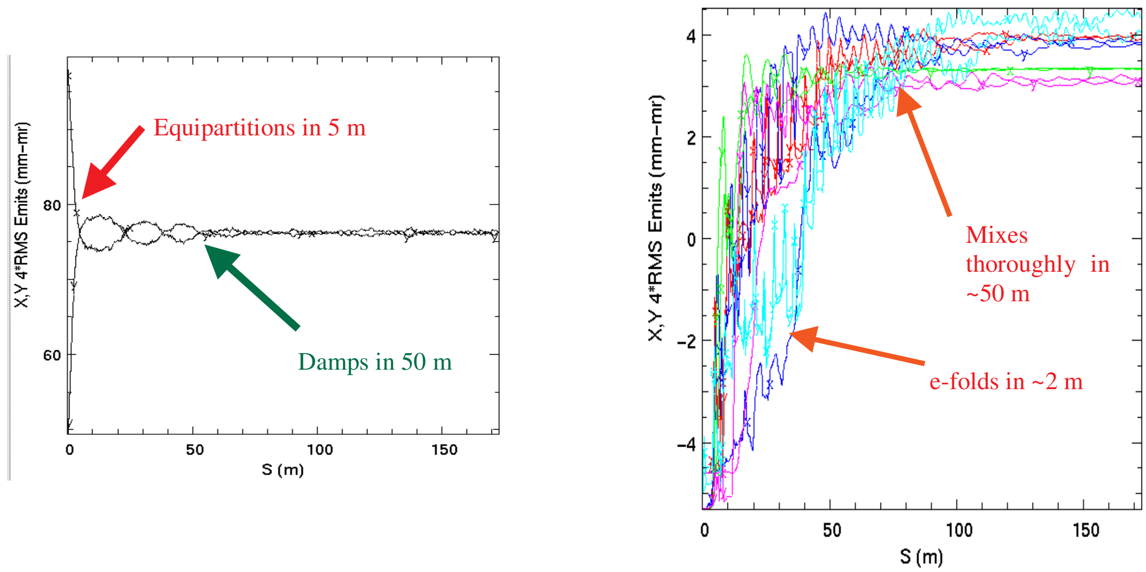

Figure 1. Equipartitioning in a cylindrical beam matched to a uniform transport channel. Beam parameters are 10 keV energy, 100 mA dc current, 1 cm radius, and 0.13 space-charge tune. The left panel shows *x* and *y* emittances versus position from source. The right panel shows the evolution of the natural logarithm of the "emittance" moment of representative ensembles, reflecting exponential growth and saturation over global portions of phase space.

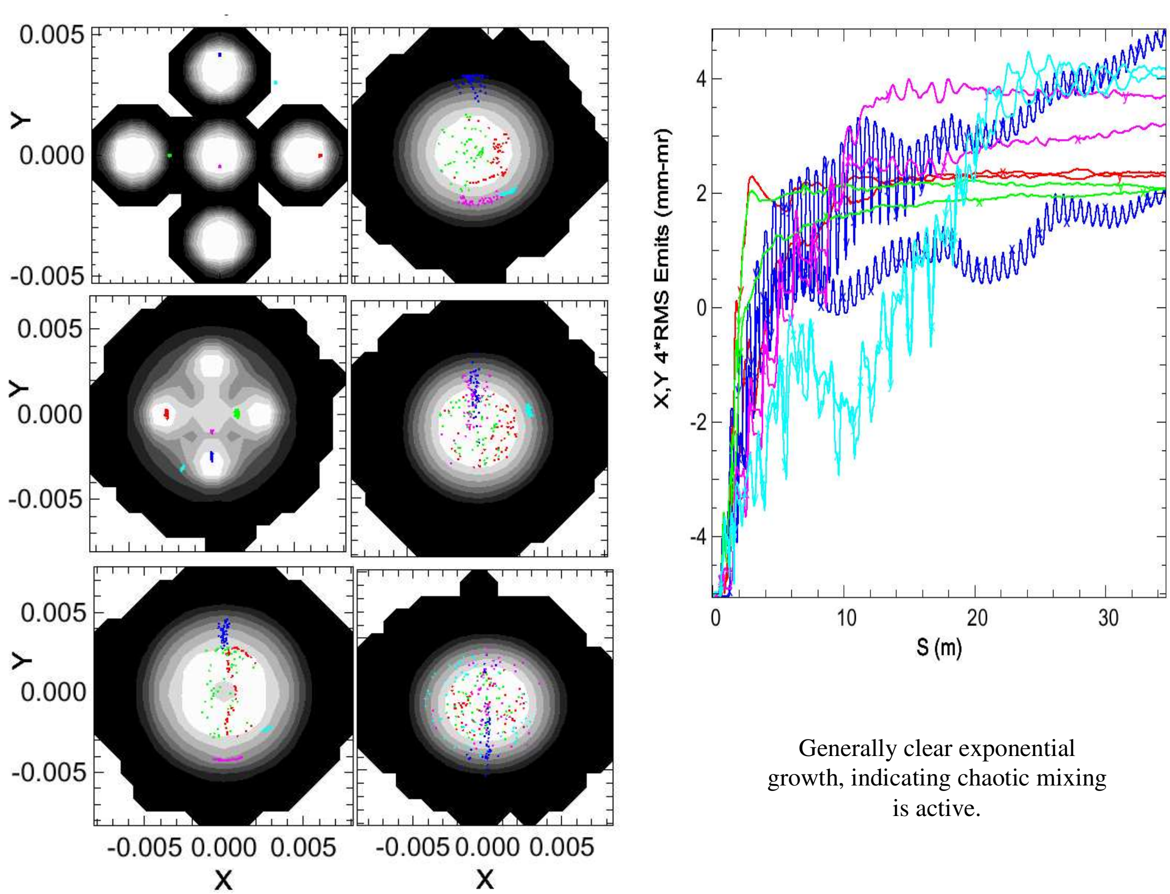

Figure 2: Evolution of five representative ensembles of test particles in the five-beamlet simulation. Beam parameters are: 5 keV energy, 44 mA current, 4.6 mm radius, and 64.8 μm full (90%) emittance. The left panel shows snapshots of the ensembles at (top-to-bottom left column) 0 m, 0.98 m, 2.88 m and (top-to-bottom right column) 5.24 m, 11.52 m, 31.68 m. The right panel shows the evolution of the natural logarithm of the *x* and *y* "emittance" moments of the ensembles. Note that the cyan ensemble is "artificial" in that it begins outside of the real beam. With the exception of the cyan ensemble, the early-time evolution of all of the ensembles is exponential.

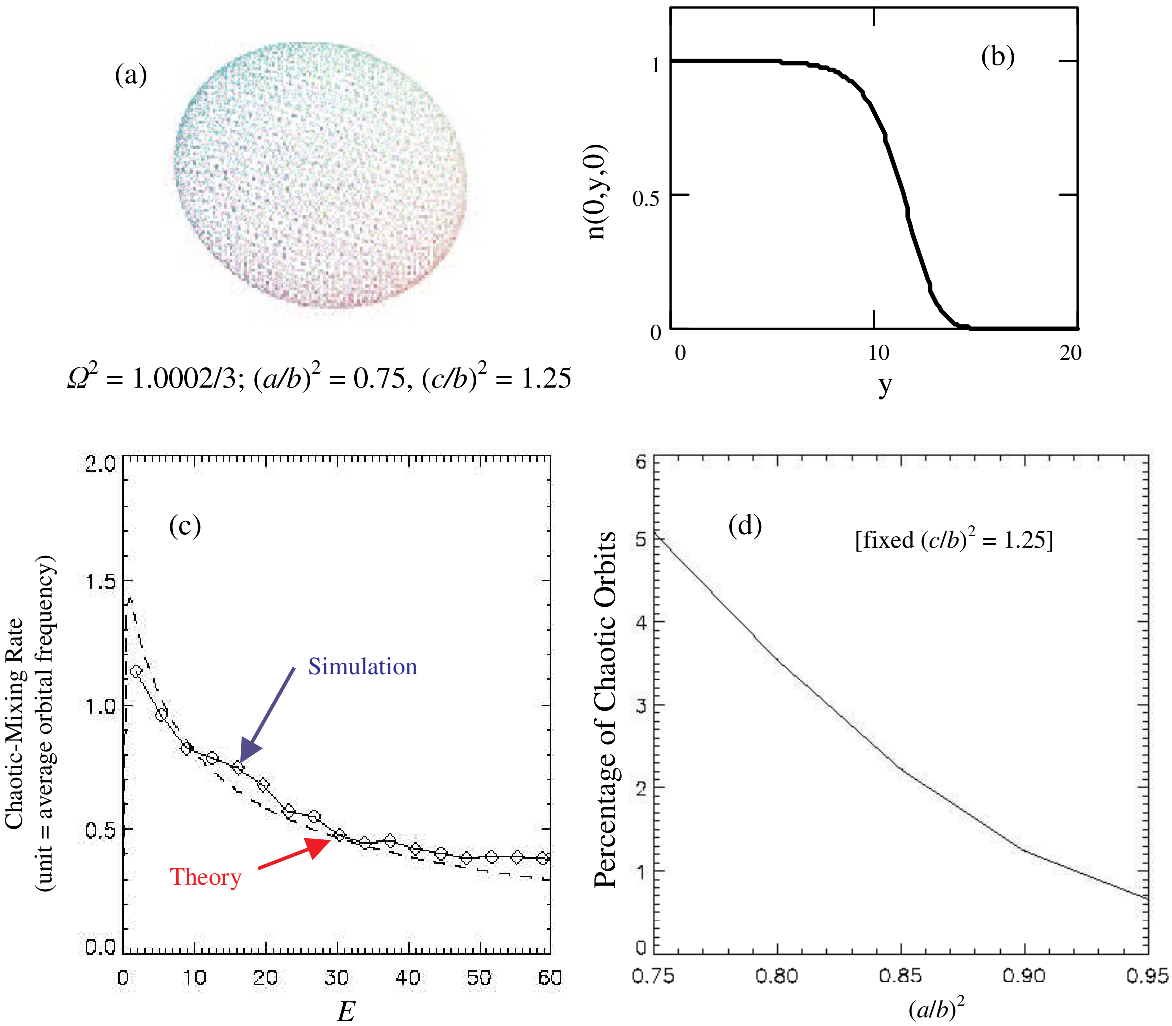

Figure 3. Results of a study of orbits in an example thermal-equilibrium bunch. (a) A representation and parameters of the coarse-grained configuration. (b) Plot of density along the *y*-axis showing the near-uniform central region and the Debye fall-off. (c) Plot of largest Lyapunov exponent, normalized to the average orbital frequency, versus total particle energy *E*. (d) Plot of the percentage of orbits that are chaotic for fixed $(c/b)^2 = 1.25$, out of all orbits that start at zero velocity within $11 < r < 25$ (corresponding to $0.5 < E < 60$), versus the square of the scale-length ratio a/b.

generally anisotropic in a reference frame comoving with the beam. We are working toward quantifying the relationship between the anisotropy of the beam and the population of chaotic orbits.

The methodology for exploring these systems computationally is to integrate orbits that start form a very close distance in phase space, i.e., by placing them with zero initial velocity at nearby points in configuration space. Because the coarse-grained net force in these systems is conservative, the total particle energies *E* are conserved. The integration proceeds for about one dynamical time, at which point the Lyapunov exponent is calculated from the particle separations. The integration is then "renormalized" to bring the orbits close again, and the process is repeated until the Lyapunov exponents converge, which typically corresponds to a duration of ~200 dynamical times.

One example is provided in Fig. 3 above. Upon expressing all lengths in units of the Debye length as measured at the bunch centroid, and all times as the product of ω_{po} with the real time *t*, one obtains the dimensionless potential-density pair

$$\Phi(\boldsymbol{x}) = (\Omega^2/2)[(b/a)^2 x^2 + y^2 + (b/c)^2 z^2] + \Phi_{sc}(\boldsymbol{x}), \qquad (2)$$
$$n(\boldsymbol{x}) = \exp[-\Phi(\boldsymbol{x})];$$

wherein Ω and (a,b,c) denote the strength and scale lengths, respectively, of the external focusing field, and $\Phi_{sc}(\boldsymbol{x})$ is the space-charge potential. Fig. 3 pertains to a triaxial configuration corresponding to $\Omega^2 = 1.0002/3$; $(a/b)^2 = 0.75$, $(c/b)^2 = 1.25$. The results reflect statistics from large, ~2000-particle, samplings of orbits that were started at zero velocity at various points in configuration space (corresponding to various total particle energies *E*). Plotted in Fig. 3c is the largest Lyapunov exponent, i.e., the chaotic-mixing rate of these orbits, normalized to the dynamical frequency. The theoretical result of Eq. (1) well matches the numerical result. Though this particular potential does admit chaotic orbits, they constitute only a modest 5% of all of the sampled orbits. However, of all orbits that reach into the density drop-off ($9 < r < 14$; cf. Fig. 3b), about 9.5% are chaotic, defined as having mixing rates in excess of 0.1 dynamical frequency. As the configuration is shaped to be more and more axisymmetric, the percentage of chaotic orbits decreases as shown in Fig. 3d. The converse turns out also to be true. However, even in a cylindrically symmetric ($a = b$) but more prolate configuration for which, e.g., $(c/b)^2 = 2$, 25% of the orbits that reach into the density fall-off are chaotic. This is a significant result, in that density perturbations arising from irregularities in the external force will tend to appear in the Debye tail, and the sizeable percentage of chaotic orbits will work toward irreversibly mixing these perturbations away. Work is in progress to quantify the entire parameter space of the potential in Eq. (2), as well as to look at the impact of closely related nonequilibrium configurations on the population of chaotic orbits.

4 SUMMARY AND FUTURE WORK

We have been probing the microscopic, i.e., orbital, dynamics in space-charge potentials. Specifically, we have presented a study of phase mixing, and have paid special attention to phase mixing of chaotic orbits for which the associated macroscopic dynamics is both irreversible and typically rapid. For emittance compensation to succeed, it needs to be completed before irreversible phase mixing has evolved substantially.

The question we have posed is: What conditions lead to a significant population of chaotic orbits in a space-charge potential? By "chaotic", we mean orbits that exponentially fill their accessible phase space -- they are "globally" chaotic, or "wildly" chaotic. We have sought to answer this question by studying both nonequilibrium beams and beams that are in thermal equilibrium. Our results thus far suggest the notion that chaotic orbits are common in space-charge configurations that are out of equilibrium, and they are present in both nonequilibrium and equilibrium configurations to a degree that tends to increase with increasing asymmetry.

What has been described herein is work in progress. Other aspects of the question that we intend to consider involve the influence of "noise", both white noise and colored noise, on the population of chaotic orbits. The motivation is simple: If there is a significant population of chaotic orbits, then it will be responsible for washing out irregularities in a fast, irreversible fashion.

A few words are in order to contrast our work with more conventional approaches based on modal analysis. When doing a mode decomposition of an inherently granular many-particle system, one is in effect imposing a prescription for coarse-graining the distribution, and is thereby focusing on macroscopic properties of the system. In a fully self-consistent paradigm, the modes will evolve according to how the particle orbits behave. If one or more modes are unstable, the instability manifests itself in the migration of particles through phase space. In the modal analysis the migration is "smoothed out" because the distribution is coarse-grained. By contrast, chaotic mixing (more generally, phase mixing) is linked directly to the behaviour of the particle orbits themselves. So a study of phase mixing enables one to pinpoint how the particles migrate -- the qualitative behaviour, time scales, etc. -- something one cannot do with a mode decomposition that describes only macroscopic behaviour of the coarse-grained system. As concerns the evolution of the overall phase space, one will get the same answer with either approach provided the modal coarse-graining is sufficiently accurate to represent the essential details (and this is the key question that confronts all modelers in that some form of coarse-graining is necessarily inherent to all simulation tools).

In summary, the phenomena of "instabilities" and phase mixing must be inextricably linked. Mode evolution tells us what is going on macroscopically, whereas mixing tells us what is going on microscopically. If one were to concentrate on the evolution of the Klimontovich distribution, which is the only valid distribution in principle, then one would be directly studying the dynamics of mixing rather than modes.

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