Dynamics of Cuspy Triaxial Galaxies with a Supermassive Black Hole

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Abstract. This talk provides a progress report on an extended collaboration which has aimed to address two basic questions, namely: Should one expect to see cuspy, triaxial galaxies in nature? And can one construct realistic cuspy, triaxial equilibrium models that are robust? Three technical results are described: (1) Unperturbed chaotic orbits in cuspy triaxial potentials can be extraordinarily sticky, much more so than orbits in many other three-dimensional potentials. (2) Even very weak perturbations can be important by drastically reducing, albeit not completely eliminating, this stickiness. (3) A simple toy model facilitates a simple understanding of why black holes and cusps can serve as an effective source of chaos. These results suggest that, when constructing models of galaxies using Schwarzschild's method or any analogue thereof, astronomers would be well advised to use orbital building blocks that have been perturbed by 'noise' or other weak irregularities, since such building blocks are likely to be more nearly time-independent than orbits evolved in the absence of all perturbations.

1. Unperturbed chaotic orbits tend to be extremely sticky

The results reported here derive from a numerical analysis of orbits in the triaxial Dehnen potentials, where (*cf.* Merritt & Fridman 1996)

$$\rho(m) = \frac{(3-\gamma)}{4\pi abc} m^{\gamma}(1+m)^{-(4-\gamma)}, \tag{1}$$

with $m^2 = x^2/a^2 + y^2/b^2 + z^2/c^2$, assuming fixed axis ratios $c/a = 1/2$ and $(a^2 - b^2)/(a^2 - c^2) = 1/2$ but allowing for a variable cusp index $0 \leq \gamma \leq 2$ and a variable black hole mass $0 \leq M_{BH}/M_{gal} \leq 10^{-2}$.

Different segments of the same chaotic orbit can be extremely different in terms of their visual appearance and their degree of exponential sensitivity (Siopis & Kandrup 2000). These differences can be quantified in terms of the sizes of short time Lyapunov exponents (*cf.* Kandrup & Mahon 1994) or the *complexity* of their Fourier spectra, *i.e.,* the degree to which the power in an orbit is concentrated near a few special frequencies (*cf.* Kandrup, Eckstein, & Bradley 1997, Siopis, Eckstein, & Kandrup 1998). Chaotic orbits typically have continuous spectra, but when they look 'nearly regular' most of the power is concentrated near a few special frequencies. In agreement with intuition, there is a strong correlation between the complexity of an orbit segment and the value of its largest short time Lyapunov exponent: chaotic orbit segments which look 'nearly regular' and have less complex spectra tend also to exhibit comparatively small exponential sensitivity.

If two chaotic orbits in the same connected phase space region are integrated for a sufficiently long time, it appears that they will eventually share the same statistical properties. However, the time required for this can be extremely long. If, for example, a single chaotic initial condition is integrated into the future, it can take as long as $100,000$ dynamical times t_D, or even longer, before the short time Lyapunov exponent exhibits a reasonable convergence towards the true Lyapunov exponent χ, as defined in a $t \to \infty$ limit. The overall rate of convergence can be quantified through an examination of distributions of short time Lyapunov exponents, $N[\chi(\Delta t)]$, generated for ensembles of chaotic orbit segments of varying length Δt. In the absence of any significant stickiness, the dispersion associated with $N[\chi]$ scales as $\sigma_\chi \propto (\Delta t)^{-p}$ with $p \approx 1/2$. For very sticky orbits, $p \ll 1/2$.

Because chaotic orbits are so sticky, one might anticipate that they could be used as building blocks for the construction of self-consistent near-equilibria which, albeit not strictly time-independent, behave as nearly time-independent entities over time intervals long compared with t_H, the age of the Universe. (In the language of Merritt & Fridman [1996], these would be 'quasi-equilibria' involving stochastic building blocks that are only 'partially mixed.') However, this supposition relies crucially on the assumption that the statistical properties of chaotic orbit segments are relatively insensitive to the effects of weak perturbations of the form which act on real galaxies. In point of fact, this does not appear to be the case.

2. Chaotic orbits can be surprisingly susceptible to very weak perturbations

Orbits in the unperturbed triaxial Dehnen potential were perturbed to mimic various effects to which real stars in real galaxies are typically exposed. Discreteness effects, *i.e.,* gravitational Rutherford scattering between individual stars, were modeled as dynamical friction and white noise, *i.e.,* near-instantaneous kicks. The effects of one or two companion objects or satellite galaxies were modeled as nearly periodic perturbations. The effects of a dense cluster environment were modeled as coloured noise, *i.e.,* random kicks of finite duration. Internal oscillations of the form that might, *e.g.,* be triggered by a close encounter were treated as a superposition of normal, or pseuo-normal, modes that induced a periodic driving and an incoherent combination of more irregular excitations modeled as coloured noise.

The basic conclusion of this investigation (Siopis & Kandrup 2000, Kandrup & Siopis 2000), consistent also with analyses of motions in other two- and three-dimensional potentials (Pogorelov & Kandrup 1999, Kandrup, Pogorelov, & Siopis 2000), is that low amplitude irregularities can have a surprisingly large effect both

- by accelerating diffusion within a given nearly disjoint phase space region; and
- by accelerating diffusion along an Arnold web or through cantori connecting nearly disjoint chaotic phase space regions.

Some of the topological obstructions associated with the Arnold web are extremely robust, so that weak perturbations have a comparatively minimal effect. However, in general such perturbations tend to accelerate dramatically the rate of phase space transport throughout the entire chaotic phase space.

The perturbations act via a resonant coupling between the characteristic frequencies of the perturbations and the frequencies of the orbits. That periodic driving works in this way should be obvious. That noise also involves a resonant coupling can be understood if one recalls (*cf.* van Kampen 1981) that a superposition of periodic forces combined with random phases is equivalent mathematically to (in general coloured) noise with a nonzero autocorrelation time t_c.

Within a nearly disjoint phase space region, the perturbations allow microscopic motions which, in a strictly time-independent potential, are prohibited by Liouville's Theorem. Thus, *e.g.,* it becomes possible for phase space trajectories to cross, which helps a collection of orbits to 'fuzz out' on short scales. The perturbations facilitate diffusion through cantori or along the Arnold web by 'jiggling' orbits in such a fashion as to help them find phase space holes.

The details of the perturbation appear largely immaterial: all that seems to matter is the amplitude of the perturbations and their characteristic time scales. Even the dependence on amplitude and time scale is comparatively weak. This implies that the details associated with realistic perturbations which might be difficult to extract from observations are largely irrelevant. The overall efficacy of the perturbations scales logarithmically in the amplitude. For time scales $t_c \gg t_D$ the perturbations have almost no effect (adiabatic limit). For somewhat shorter time scales, the dependence on t_c is again logarithmic.

But what amplitude is required to have a significant effect, *e.g.,* by destabilising nearly time-independent building blocks? Very weak white noise corresponding to relaxation times $t_R \sim 10^6 - 10^7 t_D$ can have appreciable effects within a time as short as $100 t_D$, a period which, in the inner regions of a cuspy triaxial galaxy would be short compared with t_H. Alternatively, coloured noise and/or periodic driving corresponding to perturbations of fractional amplitude as small as 10^{-3} and a characteristic time scale t_c as long as $10 t_D$ can prove important on a time scale $\sim 100 t_D$. Making t_c shorter facilitates a stronger resonant coupling between the perturbation and the orbits, thus making the perturbation even more effective.

These results suggest the possibility that galaxies could settle down towards quasi-stationary states which, albeit not true collisionless equilibria, could exist as nearly time-independent entities for times $\gg t_H$, at least in the absence of irregularities. This seems especially likely, given the recognition that, for triaxial systems, true equilibria will in general be substantially more complex than the equilibria associated with spherical and axisymmetric configurations. For a generic triaxial system, there is only one global integral, namely the energy E or Jacobi integral E_J, but it well known that equilibria $f(E)$ and $f(E_J)$ cannot be used to model triaxial systems with a strong central condensation. Unless the system is assumed to be characterised by a very special potential, *e.g.,* an integrable Staeckel potential, it cannot be in a true equilibrium unless that equilibrium involves an intricate balance of 'local integrals' (Kandrup 1998). The obvious point, then, is that even if such an intricate balance is hard to achieve, the system could evolve towards an approximate balance involving nearly time-independent building blocks.

More pragmatically, these results would also suggest that, when constructing equilibria using Schwarzschild's method or any analogue thereof, it would be strongly advisable to work with an orbit library constructed from orbits that have been evolved in the presence of weak noise or some other low amplitude perturbations. Orbits evolved in the presence of such perturbations are more likely to constitute nearly time-independent building blocks and, as such, would seem less likely to be destabilised by weak irregularities associated with discreteness effects and/or a perturbing external environment.

3. Why do black holes and cusps trigger chaos?

Numerical computations demonstrate that much of the behaviour associated with chaotic orbit ensembles evolved in the triaxial Dehnen potentials – especially those associated with orbits which, in the absence of a cusp and black hole, would correspond to regular box orbits – can be reproduced by the very simple potential

$$V(x,y,z) = \frac{1}{2}(a^2x^2 + b^2y^2 + c^2z^2) \; - \frac{GM_{BH}}{\sqrt{r^2+\epsilon^2}} \equiv V_{gal} + V_{BH}, \quad (2)$$

given as the sum of an anisotropic oscillator and a Plummer potential. Orbits in this potential exhibit the same remarkable stickiness, yield comparable distributions of short time Lyapunov exponents, and again manifest a strong susceptibility towards even very weak perturbations (Kandrup & Sideris 2000).

That this simple toy model can reproduce the qualitative features of the more complicated potential (1) suggests strongly that *the results derived for chaotic orbits in the triaxial Dehnen potentials are generic for cuspy triaxial potentials.* That the potential is so simple makes it comparatively easy to understand what exactly is going on. As noted, *e.g.,* by Merritt (1998), supermassive black holes in real galaxies are seldom if ever larger than 1% the mass of the entire galaxy and the central cuspy region typically corresponds to only a small fraction of the total mass. It follows that, for a, b, and c of order unity, the physically relevant choices of V_{BH} entail $M_{BH} \ll 1$. However, the qualitative behaviour in this regime is easily understood by a combination of perturbation theory and common sense.

For $M_{BH} \ll 1$ and $\epsilon \to 0$ in eq. (2), it appears that, except for the very lowest energies (where the potential is essentially Keplerian), essentially all of the orbits are chaotic, but that they tend to behave in a nearly regular fashion nearly all of the time. As the orbit evolves it will usually find itself in a region where $|V_{BH}| \ll |V_{gal}|$, so that the potential in which it is moving is very nearly integrable and the short time Lyapunov exponents are extremely small. Occasionally, however, the orbit will move comparatively close to the center of the galaxy, so close that $|V_{BH}|$ becomes comparable to $|V_{gal}|$. When this happens the orbit feels the competing influences of two different potentials of comparable magnitude with very different symmetries and the values

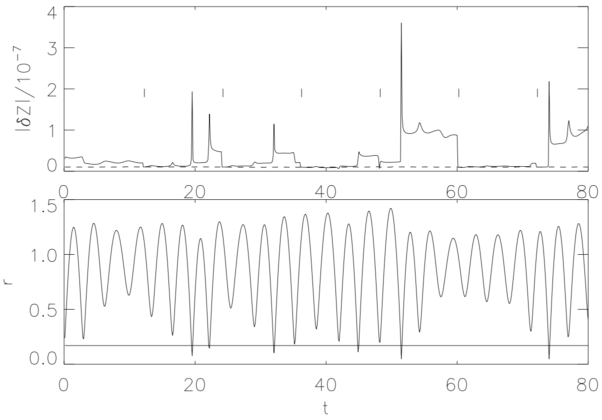

Figure 1. (a) The solid curve exhibits $|\delta Z(t)|$, the phase space distance between a perturbed and unperturbed chaotic orbit with $E = 0.75$ evolved in the potential (2) for $a^2 = 1.25$, $b^2 = 1.0$, $c^2 = 0.75$, $M_{BH} = 0.15$, and $\epsilon^2 = 10^{-4}$. The dashed line shows $|\delta Z|$ for a regular orbit evolved in the same potential with the same energy. The perturbation was renormalised at intervals $\delta t = 12$ at the points indicated by vertical stripes. (b) $r(t)$, the distance from the origin for the same unperturbed orbit at the sames times. The vertical line corresponds to $r = 0.17$, for which $|V_{BH}| \approx 0.88$.

of the positive short time Lyapunov exponents increase precipitously. The fact that, for small M_{BH}, almost all the orbits are chaotic is not difficult to understand. In the limit that $M_{BH} \equiv 0$, the orbits all reduce to boxes which densely fill a region in configuration space that includes the origin. One might expect that, for small but nonzero M_{BH}, the orbits can still pass arbitrarily close to the origin but, for any nonzero M_{BH} there is a minimum radius r_{min} inside of which $|V_{BH}|$ becomes large compared with $|V_{gal}|$.

That significant chaos is only triggered when the trajectory passes relatively close to the black hole is illustrated in Figure 1, which exhibits a segment of a chaotic orbit evolved in the potential (2). curve in the top panel exhibits the phase space separation $|\delta Z|$ between the original orbit and a perturbed orbit displaced originally by a distance $|\delta Z| = 10^{-8}$ and periodically renormalised in the usual way (*cf.* Lichtenberg & Lieberman 1992). The dashed curve shows an analogous plot of $|\delta Z|$ for a regular orbit. The lower panel plots $r(t)$, the distance from the origin. Most of the time the perturbed and unperturbed orbits remain very close together, with comparatively little systematic exponential divergence. Only when r becomes as small as ~ 0.17, so that $|V_{BH}|$ becomes as large as ~ 0.88, do the orbits tend to diverge significantly.

This toy model is particularly simple since, in the limit $M_{BH} \to 0$, all the orbits are regular boxes. For generic triaxial potentials, in the absence of a cusp or black hole one would expect both centrophilic box orbits and centrophobic tubes. Because the tubes are centrophobic, they should not in general be impacted all that much by the introduction of a cusp or a central black hole. What, *does*, however, seem to be true is that many of the orbits which, in the absence of a cusp, behave as regular boxes can, in the presence of a cusp or black hole, be converted into orbits which, albeit formally chaotic, behave in a nearly regular fashion much of the time.

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