Energy Loss of a High Charge Bunched Electron Beam in Plasma: Analysis

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There has been much recent experimental and theoretical interest in blowout regime of plasma wakefield acceleration (PWFA), which features ultra-high accelerating fields, linear transverse focusing forces, and nonlinear plasma motion. While the acceleration and focusing experienced by the beam in this regime are well understood through analytical models and tools, the same cannot be said of the plasma electron response to the beam. A quantitative understanding of the blowout regime has, to this point, been available only through detailed simulations. This paper represents an initial step towards an analytical theory of this regime, in which the energy loss of the drive beam is investigated. Using an exact analysis, we examine here a fundamental limit of nonlinear PWFA excitation, by an infinitesimally short, relativistic electron beam. The beam energy loss in this case is shown to be linear in charge even with a nonlinear plasma response, where a normalized, unitless charge exceeds

unity, and fully electromagnetic, relativistic plasma effects become important or even dominant. The physical bases underlying this persistence of linear response, which include a snow-plow effect that produces a high plasma electron density in the beam region, are pointed out. The implications of these results for the limiting case of the point charge are discussed. The analytical approach is verified through numerical solution of the fluid equations; further illumination of the physical effects deduced from the analysis is obtained through particle-in-cell simulations, and discussed in a companion paper.

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I. INTRODUCTION

The transfer of energy from short, intense electron beams to collective electron plasma oscillations is a critical component of the advanced, high-gradient acceleration scheme known as the plasma wakefield accelerator (PWFA)[1–4]. While the original proposal for the PWFA and related concepts was in the linear regime[1, 2], where the plasma oscillations can be considered small perturbations about an equilibrium, highly nonlinear regimes have recently been favored[3]. For example, in the highly nonlinear "blow-out" regime[4], the plasma electrons are ejected from the channel of the intense driving electron beam, resulting in an electron-rarefied region with excellent quality electrostatic focusing fields, as well as longitu-

dinal electromagnetic fields, which can, in tandem, stably accelerate and contain a trailing electron beam. While many aspects of the beam dynamics in this regime are linear and analytically tractable, the plasma dynamics are not. Most quantitative predictions concerning plasma behavior in the blow-out regime have been deduced from numerical simulations.

This paper represents a step in the direction of an analytical understanding of the plasma response in the blow-out regime - we shall see that new, surprising physical aspects of the beam-plasma interaction become apparent from the analysis presented. The analytical results deduced are exact in the limit of an infinitesimally short beam. Particle-in-cell simulations that both support the results given in this paper, and extend the analysis of energy loss and gain in the nonlinear PWFA to finite-length beams, are presented in an accompanying paper.

We now review some relevant background concerning the blow-out regime of the PWFA. Despite the lack of analytical models for the nonlinear plasma response, it has been noted in a number of studies [5–8] that the beam energy loss rate in the PWFA blow-out regime obeys a scaling law usually associated with the interaction of charged particles with linear media [10] (*i.e.* Cerenkov radiation). This scaling, which persists even when the beam is much denser than the plasma, and the concomitant plasma response is nonlinear and relativistic, predicts that the energy loss rate is proportional to the square of the plasma frequency [10], ω_p. Since the efficient excitation of an oscillatory system by a pulse occurs when the pulse is short compared with the oscillator period [5–8], this scaling further implies that the PWFA driving beam's energy loss rate is proportional to the inverse square of the achievable driving

beam's rms pulse length, σ_z. This prediction has led to a number of experiments that employ bunch compressors in order to decrease σ_z, thus dramatically increasing the transfer of beam energy to the plasma. In recent measurements with compressed beam at FNAL [11], the trailing portion of a 5 nC, 14 MeV, $\sigma_z = 1.2$ mm, beam pulse was nearly stopped in 8 cm of $n_0 \simeq 10^{14}$ cm^{-3} plasma, a deceleration rate of over 150 MeV/m.

The large collective field observed in this as well as other recent PWFA experiments [11, 12], was obtained the context of nonlinear plasma electron motion. Because of the onset of experiments in the nonlinear blow-out regime, the issue of wakefield scaling validity has taken on new urgency. In addition, it has recently been a proposed (in the SLAC E-164 experiment) to use ultra-short, high charge beams to a drive PWFA in the tens of GeV/m range, for creation of an ultra-high energy plasma accelerator, the so-termed "afterburner" concept[8, 13, 14]. As we shall see below, in the limit of an infinitesimally short beam, when either the plasma density or the beam charge is increased, the response of the plasma to the beam eventually becomes nonlinear. Nevertheless, the energy loss rate, somewhat surprisingly, still scales linearly with the charge in the short beam limit, even for nonlinear plasma motion. This paper is primarily intended to address the new aspects of the underlying physics of the linear-like wakefield scaling of relativistic beam energy loss in plasma. Deviations from this scaling will be studied in the accompanying simulation-based paper[9].

II. DIMENSIONLESS COLD FLUID ANALYSIS

To examine the physics relevant to ultra-short electron-beam energy loss in a plasma, we perform an analysis of the motion of cold plasma electrons having an initial ambient density n_0 (equal and opposite in charge density to the ions, which are assumed stationary) as they are excited by the passage of the ultra-relativistic ($v_b \simeq c$) beam. The state of plasma motion is described in terms of the velocity $\vec{v}$ and related momentum $\vec{p} = \gamma m_e \vec{v}$, where the Lorentz factor $\gamma = [1 - (\vec{v}/c)^2]^{\frac{1}{2}}$. The necessary relations for describing the cold plasma electrodynamic response are the Maxwell equations

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \text{ and } \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \tag{1}$$

the Lorentz force equation in convective form,

$$\frac{\partial \vec{p}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{p} = -e\left[\vec{E} + \frac{1}{c}\vec{v} \times \vec{H}\right] \tag{2}$$

and the equation of continuity for charge and current density

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{v}) = 0. \tag{3}$$

The results of our analysis will be made more transparent by the adoption of unitless variables. The natural variables used in discussing a cold plasma problem parameterize time and space in terms of the plasma frequency $\omega_p = \sqrt{4\pi e^2 n_0/m_e}$, and wave-number $k_p = \omega_p/c$, respectively, densities in terms of n_0, and the amplitudes of the and fields in terms of the

commonly termed "wave-breaking limit", $E_{WB} = m_e c\omega_p/e$. In addition, all velocities and momenta are normalized to c and $m_e c$, respectively. We thus write the spatio-temporal variables, charge and current density, and electromagnetic field components in our analysis as

$$\tilde{r} = k_p r,\ \tau = \omega_p(t - z/c),\ \tilde{v}_i = v_i/c,\ \tilde{p}_i = p_i/m_e c, \tag{4}$$

$$\tilde{n} = n_e/n_0,\ \tilde{J}_i = J_i/n_0 c \tag{5}$$

$$\tilde{E}_i = E_i/E_{WB},\ \tilde{H}_i = H_i/E_{WB}. \tag{6}$$

Note that use of Eqs. 4-6 implies that we are assuming a steady-state response (wave *ansatz*), where t and z occur only in the combination $t - z/c$. As we are eventually only interested in the region directly in contact with the relativistic beam, this steady-state assumption is entirely reasonable – it implies that we expect the energy loss rate in the plasma to be constant after the passage of a short (k_p) transient region associated with the entrance into the plasma, or some other boundary. This assumption is validated by simulations.

Using the variables in Eqs. 4-6, we may write a general equation for the azimuthal component of $\tilde{H}$

$$\frac{\partial^2 \tilde{H}_\phi}{\partial \tilde{r}^2} + \frac{1}{r}\frac{\partial \tilde{H}_\phi}{\partial \tilde{r}} - \frac{\tilde{H}_\phi}{\tilde{r}^2} = \frac{\partial \tilde{J}_r}{\partial \tau} + \frac{\partial \tilde{J}_z}{\partial \tilde{r}}. \tag{7}$$

In addition to the governing equation for $\tilde{H}$, we will need relationships between fields and current sources,

$$\frac{\partial \tilde{E}_z}{\partial \tilde{r}} = \tilde{J}_r \text{ and } \frac{\partial}{\partial \tau}\left(\tilde{E}_r - \tilde{H}_\phi\right) = -\tilde{J}_r. \tag{8}$$

In this analysis the induced is found most directly by determining the transverse current, as is customary in media-stimulated radiation calculations (*cf.* Jackson, Ref. [10]).

III. LINEAR PLASMA RESPONSE

Equation 7 is nonlinear, but may be simplified by assuming small amplitude response, in which the $|\tilde{n}|$, $\left|\tilde{E}_i\right|$ and $\left|\tilde{H}_i\right|$ are small compared to unity. In fact, to place our results in perspective, we must begin with a review of previous work in the linear regime[1, 2], which, because they are models that simplify the physical scenario, did not need to employ the induced magnetic field as the initially-solved variable.

From the viewpoint of the fluid equations, linearity importantly implies that the plasma electron response is nonrelativistic, *i.e.* $\vec{v} = \vec{p}/m_e$. In addition, the plasma electrons can then be assumed to be unaffected by magnetic fields, and the resultant fields excited behind the beam are approximately electrostatic. Note, in contrast, that in the nonlinear blowout regime, the excited fields in the rarefied region behind the beam head are qualitatively different than those in the linear, electrostatic regime. In the blowout regime, the excited fields can be described as a superposition of a radial electrostatic field due to the ions, and a TM electromagnetic wave arising from the plasma electron motion[4]. In addition, it should be noted that in deriving Eq. 7, it is also implicitly assumed that the beam density is smaller than the plasma density, or the magnetic force will not be ignorable in determining the plasma electron response (the right-hand-side of Eq. 7).

In order to illustrate the dependence of $\tilde{E}_z$ on transverse beam size, and to examine the limit of a point charge, we assume a disk-like beam, uniform up to radius $\tilde{a} = k_p a \ll 1$, and δ-function in τ. We are interested in the instantaneous response of the plasma directly upon beam passage, and integrate over the δ-function in Eq. 8 to obtain $\tilde{H}_\phi = \tilde{E}_r$, as a valid condition before, during, and immediately after beam passage. Further, we use this condition to find

$$\frac{\partial^2 \tilde{H}_\phi}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{H}_\phi}{\partial \tilde{r}} - \frac{\tilde{H}_\phi}{\tilde{r}^2} - \tilde{H}_\phi = \frac{\tilde{Q}}{\pi \tilde{a}^2}\delta(\tau)\delta(\tilde{r} - \tilde{a}). \tag{9}$$

Note that $\tilde{H}_\phi = \tilde{E}_r = 0$ just in front and behind the beam, while at $\tau = 0$ these fields are singular (they are δ-functions, while the plasma $\tilde{J}_r$ remains finite in Eqs. 8) in the ultra-relativistic $v_b \Rightarrow c$ limit. These comments are equally valid when the assumptions needed for a linearized, small amplitude analysis are violated. Note also that per Eqs. 8, the condition $\tilde{H}_\phi = \tilde{E}_r$ does not generally hold further than an infinitesimal distance behind the beam - it does not apply to the wake-region that is not in direct contact with the disk-like beam.

In Eq. 9 we have introduced a fundamental quantity that controls the scale of the beam-plasma interaction, the normalized beam charge

$$\tilde{Q} = 4\pi k_p r_e N_b. \tag{10}$$

When $\tilde{Q} \ll 1$, this indicates that the response of the system is linear. Note also that $\tilde{Q}$ can be written as $\tilde{Q} = N_b k_p^3/n_0$, which is the ratio of the number of beam electrons to plasma electrons within a cubic plasma skin-depth k_p^{-3}. Thus we may also write the underdense condition as $n_b/n_0 = \tilde{Q}/(2\pi)^{3/2} k_p \sigma_z (k_p \sigma_r)^2$, and when is near unity, and (as is typical in

previous experiments) $k_p\sigma_r \ll 1$ then $\tilde{Q} \simeq 1$ implies that the beam is denser than the plasma. It should be noted in this regard that the experiments[11, 12] that are performed in the blowout regime (n_b/n_0 well in excess of unity) have beam-plasma systems yielding $\tilde{Q}$ values between 1.5 and 4.

Equation 9 has a temporal δ-function which we again integrate over, to obtain an inhomogeneous modified Bessel equation in $\tilde{r}$

$$\frac{\partial^2 \mathbf{H}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \mathbf{H}}{\partial \tilde{r}} - \frac{\mathbf{H}}{\tilde{r}^2} - \mathbf{H} = \frac{\tilde{Q}}{\pi \tilde{a}^2}\delta(\tilde{r} - \tilde{a}), \tag{11}$$

where $H = \int_{\epsilon-}^{\epsilon+} \tilde{H}_\phi d\tau = \int_{\epsilon-}^{\epsilon+} \tilde{E}_r d\tau$. In the linear, non-relativistic limit, We interpret $\mathbf{H}$ as the total radial momentum impulse $\tilde{p}_r$, which, also in this limit, is approximately equal to $\tilde{J}_r$ immediately behind the beam. The solution to Eq. 11 is given by

$$\mathbf{H}(\tilde{r}) = \frac{\tilde{Q}}{\pi \tilde{a}} \begin{cases} K_1(\tilde{a}) I_1(\tilde{r}) & (\tilde{r} < \tilde{a}) \\ K_1(\tilde{r}) I_1(\tilde{a}) & (\tilde{r} > \tilde{a}) \end{cases} \tag{12}$$

where I_1 and K_1 are modified Bessel functions. We are interested in $\tilde{E}_z$ directly behind the beam, which is found by integrating Eq. 12

$$\tilde{E}_z(\tilde{r})\Big|_{\tau=\epsilon+} = \int_\infty^{\tilde{r}} \mathbf{H}(\tilde{r}')d\tilde{r}' = \frac{\tilde{Q}}{\pi \tilde{a}^2} \begin{cases} 1 - \tilde{a} K_1(\tilde{a}) I_0(\tilde{r}) \\ \tilde{a} I_1(\tilde{a}) K_0(\tilde{r}) \end{cases} \tag{13}$$

over the radial coordinate.

In the limit that $\tilde{a} \ll 1$, the field inside of the disk region is nearly constant, and given by

$$\tilde{E}_z(\tilde{r})\Big|_{\tau=\epsilon+} \simeq \frac{\tilde{Q}}{\pi \tilde{a}^2}[1 - \tilde{a} K_1(\tilde{a})] \simeq \frac{\tilde{Q}}{2\pi}\left[\ln\left(\frac{2}{\tilde{a}}\right) - 0.577\ldots\right] \tag{14}$$

which is to leading order proportional to $\tilde{Q}/2\pi$. In physical units we may write Eq. 14 as

$$eE_z|_{\tau=\epsilon+} \simeq 2e^2 k_p^2 N_b \ln\left(\frac{1.123}{k_p a}\right). \tag{15}$$

Several comments arise from inspection of Eq. 15. The first is that the scaling of E_z with respect to wavenumber k is dominated by the factor k_p^2 that is typical of Cerenkov radiation [10], if we interpret k_p as the maximum allowable value of k that is radiated (in the linear regime it is the only value). The second comment is that the linear result is ill-behaved in the limit of $k_p a \ll 1$, as Eq. 15 predicts a logarithmic divergence in E_z. This pathology is a result of allowing $\tilde{J}_r$ (through $\mathbf{H}$) to diverge as r^{-1}.

In the limit of $a \to 0$ (the point-charge limit), this divergence is mitigated by the replacement of a in Eq. 15 with the minimum impact parameter, b_{min}, using either the classical ($b_{min} = e^2/\gamma m_e$) or quantum mechanical (see discussion at end of next section) expression for this parameter. In this limit, a significant portion of the energy loss predicted by equation 15 is due to small impact parameter collisions with a correspondingly large energy transfer. These particles are energetic enough to simply leave the plasma region, and are thus unable to fully couple their energy back to the plasma wave.

Previous analyses by Jackson[10] and Chen, *et al.*,[1] rely on arguments involving the Debye length λ_d to calculate the the point-particle energy loss limit. Although their approaches appear to have some similarity at first glance, their treatment is, in fact, quite different. In Jackson's treatment, the integral is split between impact parameters $b > 1/k_d = \lambda_p/2\pi$, and $b < 1/k_d$, where the plasma behaves as a continuous medium in the former case, while

discrete particle effects become dominant in the latter case. Away from this boundary, the contribution to the integral does not depend on the region being considered, and, indeed, Jackson's final result is independent of k_d. The fact that one integral has a sharp cutoff at $1/k_d$ while the other has an exponential decay leads to a slight under-representation of energy loss to particles with $b \simeq 1/k_d$. This small discrepancy is in line with approximations made up to that point in the calculation.

In Chen's treatment, the entire plasma is modeled using linearized, cold-fluid assumptions. Chen's solution for the wakefield is divergent, having a spike near $r = 0$ that persists for many cycles. The authors believe this to be an unphysical result, and attribute it to a lack of thermal effects in the model. They claim that field quantities derived from their result are only valid in a region where $r > \lambda_d$. Unlike in Jackson's analysis where λ_d is introduced as a screening distance, here it seems to represent the distance traveled by a thermal plasma electron in a period comparable to a plasma cycle, thus smearing out the spike on approximately that length scale. We remark that a more likely cause for the divergence is the lack of discrete particle effects in their model, which fails to consider the effects of the minimum impact parameter. The nonlinear fields close to a point-like driver will also cause some local wave-breaking, which cannot be represented in a fluid model having a single-valued velocity function. This effect will certainly modify the form of the long-range wakefield, possibly removing the divergence seen in Chen's result.

IV. NONLINEAR PLASMA RESPONSE

As $\tilde{Q}$ is raised, we must consider the plasma electrons' relativistic response to large amplitude fields, under the general condition that $\tilde{H}_\phi = \tilde{E}_r$. The problem of charges moving in perpendicular electric and magnetic fields of equal magnitude has been solved by Landau[15], and we base our analysis on this solution. In this analysis, the relation between the transverse and longitudinal momentum (in our units) is,

$$p_z = -\frac{\alpha}{2} + \frac{p_z^2 + \epsilon^2}{2\alpha}, \tag{16}$$

where α and ϵ are two constants which are both equal to unity for our initial conditions where the particle starts from rest. Equation 16 then becomes,

$$p_z = \frac{p_r^2}{2}. \tag{17}$$

This result is not dependent on the fields being constant in r or z, but only on them being orthogonal and equal.

The passage of the beam induces not only an impulsive change in the radial momentum, but a longitudinal impulse in the positive (beam motion) direction. Thus, for large $\mathbf{H}$, the plasma electrons experience a large forward momentum impulse, and may have a relativistic v_z just after passage of the beam.

The equation of motion for p_r is,

$$\frac{dp_r}{dt} = (1 - v_z)H \tag{18}$$

where the time derivative can be transformed to the beam longitudinal coordinate $\xi = z - t$ with the relation $d/dt = -(1 - v_z)d/d\xi$. Cancellation of the $1 - v_z$ terms and integration over the beam extent leads to the result

$$\tilde{p}_r = \mathbf{H}, \tag{19}$$

which is identical to the result for the linear case.

With the use of Eqs. 19 and 17 the plasma electron's transverse velocity becomes

$$\tilde{v}_r = \frac{\mathbf{H}}{\sqrt{1 + \mathbf{H}^2 + \frac{1}{4}\mathbf{H}^4}} = \frac{\mathbf{H}}{1 + \frac{1}{2}\mathbf{H}^2} \tag{20}$$

In order to relate this result to $\tilde{J}_r$ we must multiply by $\tilde{n}$, which due to the change in $\tilde{v}$ directly after passage of the beam, is predicted with the aid of the continuity relation (Eq. 3), and Eqs.19 and 17 to be

$$\tilde{n} = (1 - \tilde{v}_z)^{-1} = 1 + \frac{1}{2}\mathbf{H}^2. \tag{21}$$

Thus, we are led to the remarkable result that the relativistically correct induced radial current is identical to the approximate, linear, non-relativistic expression,

$$\tilde{J}_r = \tilde{n}\tilde{v}_r = \left(1 + \frac{1}{2}\mathbf{H}^2\right) \cdot \frac{\mathbf{H}}{1 + \frac{1}{2}\mathbf{H}^2} = \mathbf{H} \tag{22}$$

Since the induced $\tilde{J}_r$ is unchanged from the linear case, the analysis of the decelerating field $\tilde{E}_z$ leading to Eq. 13 remains valid. Therefore we see that the "linear" scaling observed in

simulations of short pulse beam-excited wake-fields may be have some basis in this on an analytically predicted effect.

The result in Eq. 22 arises from two effects which cancel each other: the induced $\tilde{v}_r$ saturates (at a value well below 1), yet the density enhancement due to longitudinal motion - a "snow-plowing" of the plasma electrons by the electromagnetic pressure - exactly makes up for this saturation, and the induced $\tilde{J}_r$ remains linear in $\tilde{Q}$. This snow-plowing is analogous to the scenario from laser wake-field acceleration, where the electromagnetic pressure in gradient of a short, intense laser gives rise to a density enhancement in the laser's leading edge.

In order for $\tilde{n}$ to be enhanced directly after the beam passage, no net longitudinal displacement of the plasma electrons while the beam interaction must occur – the density enhancement is caused only after the beam by particles with relativistic longitudinal velocity "catching up" to the beam. In fact, when one takes the limit of a δ-function in ζ in the integrals of the motion, one finds that the displacements in both r and z vanish in this limit. This is reassuring, as it validates our use of the Landau result. Numerical integration of the equations of motion also confirm our result. In this case if one performs the integration over ζ as the independent variable, care must be taken to expand the differential interaction period by the factor $(1 - \tilde{v_z})^{-1}$ as described in obtaining Eq. 19.

The effects described by Eqs. 17 and 19 can be alternatively be confirmed by performing an analysis in the rest frame of the beam and Lorentz transforming to the laboratory frame. It should be noted that the vanishing of the radial displacement of the plasma electrons during

the passage the δ-function beam is necessary for self-consistency with the assumptions of the derivation leading to Eqs.17 and 19. If the plasma electrons experience non-negligible displacement, then Eq. 19 and must be revisited. It is because we can neglect radial displacement in the δ-function beam limit that our analysis can rigorously account for radial variations in the plasma response – it is manifestly a two-dimensional axisymmetric result. In the δ-function beam limit, one may also self-consistently ignore effects of the induced longitudinal force from the plasma electrons themselves during their infinitesimally short interaction time with the beam, as this force remains finite and its integrated effect tends to zero.

The results obtained here are similar in appearance to those that have been derived in the context of the one-dimensional laser-plasma interaction by Sprangle, *et al.*[17] Because of some considerable confusion in formal discussions of the present results, we must remark on this situation. This similarity is not a result of the physical scenario, but arises because our 2D time-dependent results (r, z, t) bear a mathematical resemblance to the 1D fluid analysis undertaken in Ref.[17]. In our case and in Ref.[17], the analysis is begun with the wave, or steady state, assumption – all time dependence enters into the problem through the variable $\tau = t - z/c$. However, our analysis is explicitly 2D, in (r, τ), while in the 1D analysis laser-plasma analysis, the system reduces to simply τ. In our derivation, we reduce Eq. 7 to Eq. 11 by looking only in an infinitesimal region in around the beam, and only the radial coordinate remains as a variable. By integrating over the δ-function in τ, we finally obtain a 1D equation, but it is not in τ, it is in r. The coincidental similarities between our Eqs. 17-22

and the analogous expressions in Ref.[17] arise because in our case, during the passage of the beam, the effect of the δ-function integration on the equation of continuity enters purely through v_z and not v_r. Thus the powerful relation $\tilde{n} = (1 - \tilde{v_z})^{-1}$ is valid in both cases. In addition, because in both our case and the 1-D laser-plasma case, the phase velocity of the excitation is relativistic, the relationship $\tilde{p}_z = \frac{1}{2}\tilde{p_r}^2$ holds, and several of the relations listed in Sprangle, *et al.*, reduce to ours – assuming that the transverse momenta discussed in both cases are equivalent, which they are not, as we discuss below. These factors, and these along, are all there is to the formal similarity of our results to those of Ref.[17].

Beyond these similarities, the present analysis and that of Sprangle, *et al.*, are completely divergent, as a cursory examination indicates. The analysis we have presented is of an electron beam excitation of a plasma with the radial variation of the induced longitudinal field that arises from plasma currents driven by the beam. Sprangle's analysis, on the other hand, concerns the plasma response to a transverse wave – a laser beam, with only longitudinal variation, and which has no free charge. This is a critical point, because in Ref.[17] the transverse component of the vector potential $\vec{A}$ is introduced, whereas it is well-known that in beam-driven wake-fields, one must solve for the longitudinal component of the vector potential A_z, as is done in Refs.[1, 18]. Further, we note that the equations governing the potentials in Ref.[17] and the integrated field in our case are completely different, as one would expect because they are completely different physical scenarios. One may not, except by error, deduce the present results from those of Ref.[17].

As a way of further investigating the nonlinear response of the plasma to the beam charge,

we now examine the energy content of the excitation left in the beam's wake. The energy per unit length which must be supplied by the beam is found found by integrating the differential energy density in the plasma motion and field just behind the beam;

$$\begin{aligned}\frac{d\tilde{U}}{d\tilde{z}} &= 2\pi \int_0^\infty \left[\sqrt{1+\tilde{p}_r^2+\tilde{p}_z^2}-1\right]\tilde{n}(1-\tilde{v}_z)\tilde{r}d\tilde{r} + 2\pi\int_0^\infty \tfrac{1}{2}\tilde{E}_z^2(\tilde{r})\tilde{r}d\tilde{r} \\ &= \pi\left[\int_0^\infty H^2\tilde{r}d\tilde{r} + \int_0^\infty \tilde{E}_z^2(\tilde{r})\tilde{r}d\tilde{r}\right]\end{aligned} \tag{23}$$

Several comments are in order at this point: the first is that just behind the beam $\tilde{H}_\phi = \tilde{E}_r = 0$, so there is no contribution to the field energy density from these field components. The second is that in order to find the differential spatial rate of mechanical energy deposition by the beam, one must take into account that the plasma electrons may be traveling in z, which means that the normalized mechanical energy density $\tilde{n}(\gamma-1)$ must be multiplied by $(1-\tilde{v}_z)$, as shown in in the first integral above. This factor, which is may be understood by analogy with a familiar effect found in the study of electromagnetic wake-fields in accelerators (although not as familiar as it perhaps should be to all, see the discussion in Ref.[16]) removes the dependence of the energy loss on powers $\tilde{Q}$ of larger than 2. Evaluation of the integrals given above yields

$$\begin{aligned}\frac{d\tilde{U}}{d\tilde{z}} &\simeq \frac{\tilde{Q}^2}{2\pi\tilde{a}^2}\int_a^\infty \left[K_0^2(\tilde{r})+K_1^2(\tilde{r})\right]\tilde{r}d\tilde{r} \\ &= \frac{\tilde{Q}^2}{2\pi\tilde{a}^2}\left[1-\tilde{a}K_1(\tilde{a})I_0(\tilde{a})\right] \simeq \tfrac{1}{2}\tilde{Q}\tilde{E}_z(\tilde{r})\Big|_{\tau=\epsilon+}\end{aligned} \tag{24}$$

as expected. The factor of one-half is also familiar from the study of both plasma[2] and electromagnetic wake-fields – it arises from the averaging of the force over the bunch (zero at the front, maximum at the back), and taking the limit as the bunch length goes to zero.

As the results of Eqs. 17-20 concern beams of negligible length, they are applicable, in the limit $\tilde{a} \to 0$, to the case of a single particle. The effects of nonlinear plasma electron response do not, as might have been hoped, remove the logarithmic divergence seen in Eq. 14. Note that the logarithmic term in Eq. 15 corresponds to the familiar Coulomb logarithm[10], with an argument that is the ratio of the maximum to minimum impact parameter b, $\ln(b_{\max}/b_{\min})$. We deduce that the upper limit $b_{\max} = 2/k_p$, while the lower limit in the analysis is a. The value of a in Eqs. 13-15 cannot be drawn towards zero without violating several assumptions of our analysis, however. The fluid assumption is fine; modeling the plasma electrons as a continuous fluid introduces errors not in the average energy loss, but in the fluctuations of this quantity. For ultra-relativistic particles, quantum mechanical effects constrain the minimum impact parameter[10] to $b_{\min} \simeq (\hbar/m_e c)\sqrt{2/\gamma}$ through the uncertainty principle, however. Thus we write the energy loss rate for a point particle of charge q as

$$\frac{dU}{dz} \simeq q^2 k_p^2 \ln\left(0.794\sqrt{\gamma}\frac{m_e c}{k_p \hbar}\right) \simeq q^2 k_p^2 \ln\left(5\sqrt{\gamma}\frac{\lambda_p}{\lambda_c}\right) \tag{25}$$

where $\lambda_p = 2\pi/k_p$ and λ_c are the plasma and Compton wavelengths, respectively. Note that both limits in the Coulomb logarithm can be viewed quantum mechanically, as the minimum quantum of energy loss (emission of a plasmon) in the plasma is in fact $\hbar\omega_p$, as has been verified experimentally for very thin foils[19].

V. NUMERICAL INTEGRATION OF THE AXISYMMETRIC FLUID EQUATIONS

It is important to validate the infinitesimal length analysis of the previous sections, which by allowing an exact solution of the energy loss problem gives insight into the microscopic processes which are present in the nonlinear PWFA. Our analysis has been checked with numerical integrations of the two dimensional fluid equations for finite length beams, having a longitudinal charge distribution, $\rho_b \sim \exp(-z^2/2\sigma_z^2)$, and exploring the limit that $k_p\sigma_z \to 0$. In order to connect with the point beam limit, and to accurately quantify the energy imparted to the plasma, we compare the average on-axis beam energy loss rate, $(2\pi\sigma_z)^{-1}\int eE_z(0,\zeta)\exp(\zeta^2/2\sigma_z^2)d\zeta$ (where $\zeta = -c\tau$) for these cases with linear theory. The predictions of linear theory are obtained from using Eq. 14 to give the Green function (δ-function response), and performing a convolution integral[2] of the over the Gaussian pulse, to give an average energy loss rate of $\frac{\tilde{Q}}{2\pi\tilde{a}^2}[1-\tilde{a}K_1(\tilde{a})]\exp(-k_p^2\sigma_z^2)$. The results of these simulations are shown in Fig. 1, which displays the average energy loss of a beam in the linear regime ($\tilde{Q} = 0.002$), a comparison to linear analytical theory, and the nonlinear regime ($\tilde{Q} = 2$). In the $\tilde{Q} = 0.002$ case, the fluid simulations agree extremely well with analytical predictions. For the case with $\tilde{Q} = 2$, the simulations disagree with linear theory over a broad range of pulse lengths, but converge to the linear theory in the limit that , as expected from the conclusions we have drawn from Eqs. 13 and 19-22. Note that the numerical integration of the fluid equations is not easily stabilized when $\tilde{Q} > 2$, and thus to perform

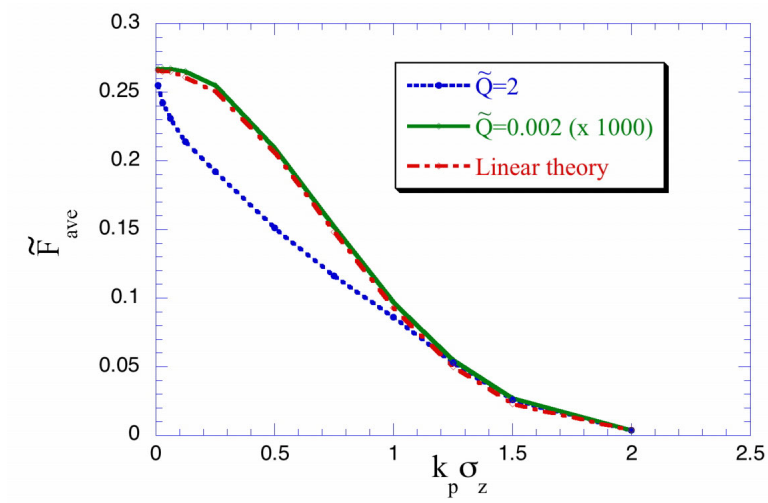

Figure 1: The average normalized energy loss rate $\tilde{F}_{dec} = e\bar{E}_z/m_e c\omega_p$ of an electron beam with $k_p a = 0.2$, as a function of $k_p\sigma_z$, for $\tilde{Q} = 0.002$ (diamonds, solid line) and $\tilde{Q} = 2$ (circles, dotted line) from cylindrically symmetric fluid simulation, and linear theory (diamonds, dashed line).

further numerical investigations another tool must be adopted. The investigations using such a tool, electromagnetic particle-in-cell (PIC) simulation, are discussed in the companion work to this paper[9]. We note their conceptual importance is found in that the results of fluid integrations rely on a model that is simply connected to our analytical results (they are only a check on the validity of the derivation), while the PIC simulations are an entirely different, more complete model of the beam-plasma interaction. The PIC codes thus give a check on the physics of our analysis results, not solely the mathematics.

VI. CONCLUSIONS

In conclusion, we restate the most surprising of our results, that the fully relativistic response of a plasma to the passage of an ultra-short beam gives an induced electric field that is identical to the linear result. We have in the process identified a single parameter, the normalized charge $\tilde{Q}$, which identifies when a bunched beam may be expected to give rise to nonlinear motion in the plasma. Further, this parameter may be used to predict the maximum physically achievable energy loss of a beam in plasma, Eq. 15. The interplay between the nonlinear effects - relativistic saturation of transverse velocity, and snow-plowing of plasma density - which cancel for infinitesimal, but not for finite, length beams must be studied in more detail by simulation. Such a study is now actively under way[9].

In addition to the verification of the relevant physical processes at play in the plasma response in finite-length beams, PIC simulations in the companion paper delve more deeply into the question of the scaling of both decelerating and accelerating wake-fields with charge ($\tilde{Q}$) in the nonlinear limit. In agreement with the results of the present analysis, the beam energy loss found in simulation scales nearly linearly with charge until $\tilde{Q}$ well in excess of unity. At high enough $\tilde{Q}$ notable deviations from linear scaling are present, and the wake-field amplitude tends to saturate. These results have serious implications for planned experimental work.

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