Energy Loss of a High Charge Bunched Electron Beam in Plasma: Nonlinear Plasma Response and Linear Scaling

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There has been much experimental and theoretical interest in blowout regime of plasma wakefield acceleration (PWFA), which features ultra-high accelerating fields, linear transverse focusing forces, and nonlinear plasma motion. Using an exact analysis, we examine here a fundamental limit of nonlinear PWFA excitation, by an infinitesimally short, relativistic electron beam. The beam energy loss in this case is shown to be linear in charge even for nonlinear plasma response, where a normalized, unitless charge exceeds unity, and relativistic plasma effects become important or dominant. The physical bases for this persistence of linear response are pointed out. As a byproduct of our analysis, we re-examine the issue of field divergence as the point-charge limit is approached, suggesting an important modification of commonly held views of evading unphysical energy loss. Deviations from linear behavior are investigated using simulations with finite length beams. The peak accelerating field in the plasma wave excited behind a finite-length beam is also examined, with the artifact of wave spiking adding to the apparent persistence of linear scaling of the peak field amplitude well into the nonlinear regime. On the other hand, at large enough normalized charge, linear scaling of fields collapses, with serious consequences for plasma wave excitation efficiency. The dramatic implications of these results for observing the collapse of linear scaling in planned experiments are discussed.

I. INTRODUCTION

The transfer of energy from short, intense electron beams to collective electron plasma oscillations, known as plasma wakefields, has been discussed in the context of the plasma wakefield accelerator (PWFA)[1-4]. While the original proposal for the PWFA and related concepts was in the linear regime[1,2], where the plasma

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oscillations can be considered small perturbations about an equilibrium, highly nonlinear regimes have recently been favored[3]. For example, in the highly nonlinear "blow-out" regime[4], the plasma electrons are ejected from the channel of the intense driving electron beam, resulting in an electron-rarefied region with excellent quality electrostatic focusing fields, as well as longitudinal electromagnetic fields, which can, in tandem, stably accelerate and contain a trailing electron beam. While the beam dynamics in this regime are linear and analytically tractable, the plasma dynamics are not, and most predictions concerning plasma behavior in the blow-out regime have been deduced from numerical simulations. This paper represents a step in the direction of an analytical understanding of the plasma response in the blow-out regime — we shall see that new, surprising physical aspects of the beam-plasma interaction become apparent from our analysis and accompanying simulations.

Despite the lack of analytical models for the nonlinear plasma response, it has been noted in a number of studies [5-8] that the beam energy loss rate in the PWFA blow-out regime obeys scaling usually associated with the interaction of charged particles with linear media [9]. This scaling, which persists even when the beam is much denser than the plasma, and the concommitant plasma response is nonlinear and relativisitic, predicts that the energy loss rate is proportional to the square of the plasma frequency [9], ω_p^2. Since the efficient excitation of an oscillatory system by a pulse occurs when the pulse is short compared with the oscillator period [5-8], this scaling further implies that the PWFA driving beam's energy loss rate is proportional to the inverse square of the driving beam's rms pulse length, σ_z. This prediction has led to a number of experiments that employ bunch compressors in order to decrease σ_z, thus dramatically increasing the transfer of beam energy to the plasma. In recent measure-ments with compressed beam at FNAL [10], the trailing portion of a 5 nC, 14 MeV, σ_z=1.2 mm, beam pulse was nearly stopped in 8 cm of $n_0 \cong 10^{14}$ cm^{-3} plasma, a deceleration rate of over 150 MeV/m. The large collective field observed in this as well as other recent PWFA experiments [10,11], was obtained the context of nonlinear plasma electron motion, thus re-opening the issue of wakefield scaling validity. In addition, it has recently been a proposed (in the SLAC E-164 experiment) to use ultra-short, high charge beams to a drive PWFA in the tens of GeV/m range, for creation of an ultra-high energy plasma accelerator [8,12,13]. This paper is primarily intended to address the underlying physics of the linear-like wakefield scaling of relativistic beam energy loss in plasma, and to study deviations from this scaling.

The typical longitudinal field profile excited in the blow-out regime is shown in Fig. 1. Three measures of the field amplitudes are given in this figure: the well-behaved decelerating field inside of the driving beam, the peak accelerating field, which is characterized by a narrow spike, and "useful" field, that which precedes the spike. This final measure is termed "useful" because the spike is an extremely narrow region, with negligible stored energy, and therefore of very limiting use for efficiently accelerating a real beam.

It will be shown that the scaling of plasma wake-field amplitudes, as measured in particular by the peak accelerating field spike, follows linear-like behavior well into the nonlinear regime. The mechanisms behind this anomalous scaling are explored, as are the ways in which they fail in the extremely nonlinear limit.

This work represents an extension of the analysis originally presented in a manuscript by Barov, *et al.,* [14], which is presently pending publication. It provides more background on this original analytical work, and significant new simulation results.

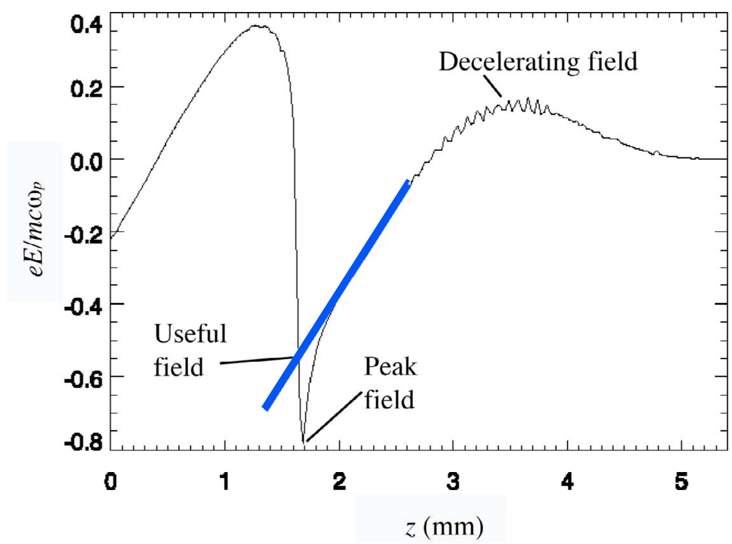

Figure 1. The longitudinal field profile given by PIC simulation for PWFA excitation in the blowout regime. The decelerating field, peak accelerating field, and a defined "useful field", which avoids the narrow spike region, are indicated in the drawing.

II. DIMENSIONLESS COLD FLUID ANALYSIS

To examine the relevant physics, we perform an analysis of the motion of cold plasma electrons with equilibrium density n_0 (equal and opposite in charge density to the ions, which are assumed stationary) as they are excited by a relativistic ($v_b \cong c$) electron beam. The state of plasma motion is described in terms of the velocity $\vec{v}$ and momentum $\vec{p} = \gamma m_e \vec{v}$, where the Lorentz factor $\gamma = \left[1 - (\vec{v}/c)^2\right]^{-1/2}$. The necessary relations for describing the cold plasma electrodynamic response are the Maxwell equations

$$\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t},\ \text{and}\ \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{H}}{\partial t},\qquad[1]$$

the Lorentz force equation in convective form,

$$\frac{\partial\vec{p}}{\partial t}+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{p}=-e\left[\vec{E}+\frac{1}{c}\vec{v}\times\vec{H}\right],\qquad[2]$$

and the equation of continuity for charge and current density

$$\frac{\partial n}{\partial t}+\vec{\nabla}\cdot(n\vec{v})=0.\qquad[3]$$

The results of our analysis will be made more transparent by the adoption of unitless variables. The natural variables used in discussing a cold plasma problem parameterize time and space in terms of the plasma frequency $\omega_p=\sqrt{4\pi e^2 n_0/m_e}$, and wave-number $k_p=\omega_p/c$, respectively, densities in terms of n_0, and the amplitudes of the $\vec{E}$ and $\vec{H}$ fields in terms of the commonly termed "wave-breaking limit" $E_{WB}=m_e c\omega_p/e$. In addition, all velocities and momenta are normalized to the c and $m_e c$, respectively. We thus write the spatio-temporal variables, charge and current density, and electromagnetic field components in our analysis as

$$\tilde{r}=k_p r,\ \tau=\omega_p\left(t-z/c\right),\ \tilde{v}_i=v_i/c,\ \tilde{p}_i=p_i/m_e c,\qquad[4]$$

$$\tilde{n}=n_e/n_0,\ \tilde{J}_i=J_i/n_0 c\qquad[5]$$

$$\tilde{E}_i=E_i/E_{WB},\ \tilde{H}_i=H_i/E_{WB}.\qquad[6]$$

Note that use of Eqs. 4 imply that we are assuming a steady-state response (wave *ansatz*), where t and z occur only in the combination $t-z/c$. With these variables, we may write a general equation for the azimuthal component of $\tilde{H}$

$$\frac{\partial^2\tilde{H}_\phi}{\partial\tilde{r}^2}+\frac{1}{\tilde{r}}\frac{\partial\tilde{H}_\phi}{\partial\tilde{r}}-\frac{\tilde{H}_\phi}{\tilde{r}^2}=\frac{\partial\tilde{J}_r}{\partial\tau}+\frac{\partial\tilde{J}_z}{\partial\tilde{r}}.\qquad[7]$$

In addition to the governing equation for $\tilde{H}$, we will need relationships between fields and current sources,

$$\frac{\partial\tilde{E}_z}{\partial\tilde{r}}=\tilde{J}_r\ \text{and}\ \frac{\partial}{\partial\tau}\left(\tilde{E}_r-\tilde{H}_\phi\right)=-\tilde{J}_r.\qquad[8]$$

In this analysis the induced $\tilde{E}_z$ is found most directly by determining the transverse current, as is customary in media-stimulated radiation calculations (*cf.* Jackson, Ref. 9).

III. LINEAR PLASMA RESPONSE

Equation 7 is nonlinear, but may be simplified by assuming small amplitude response, in which the $|\tilde{n}|$, $|\tilde{E}_i|$ and $|\tilde{H}_i|$ are small compared to unity. In fact, to place our results in perspective, we must begin with a review of previous work in the linear regime [1,2], which because they are simpler models did not need to use the induced magnetic field as the initially solved variable.

From the viewpoint of the fluid equations, linearity importantly implies that the plasma electron response is nonrelativistic, *i.e.* $\vec{v} = \vec{p}/m_e$. In addition, the plasma electrons are not affected by magnetic fields, and the resultant fields excited behind the beam are approximately electrostatic. Note that in the blowout regime, the excited fields in the rarefied region behind the beam head are qualitatively different, and can be described as a superposition of a radial electrostastic field due to the ions, and a TM electromagnetic wave arising from the plasma electron motion.

In order to illustrate the dependence of $\tilde{E}_z$ on transverse beam size, and to examine the point charge limit, we assume a disk-like beam, uniform up to radius $\tilde{a} = k_p a << 1$, and δ-function in τ. We are interested in the instantaneous response of the plasma directly upon beam passage, and integrate over the δ-function in Eq. 8 to obtain $\tilde{H}_\phi = \tilde{E}_r$ before, during, and immediately after beam passage. Further, we use this condition to find

$$\frac{\partial^2 \tilde{H}_\phi}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{H}_\phi}{\partial \tilde{r}} - \frac{\tilde{H}_\phi}{\tilde{r}^2} - \tilde{H}_\phi = \frac{\tilde{Q}}{\pi \tilde{a}^2}\delta(\tau)\delta(\tilde{r} - \tilde{a}). \qquad [9]$$

Note that $\tilde{H}_\phi = \tilde{E}_r = 0$ just in front and behind the beam, while at $\tau = 0$ these fields are singular (being δ-functions, while the plasma $\tilde{J}_r$ remains finite in Eq. 8) in the ultra-relativistic $v_b \Rightarrow c$ limit. These comments are also valid when the assumptions needed for a linearized, small amplitude analysis are violated.

In Eq. 9 we have introduced a fundamental quantity that controls the scale fo the beam-plasma interaction, the normalized beam charge

$$\tilde{Q} = 4\pi k_p r_e N_b. \qquad [10]$$

When $\tilde{Q} << 1$, indicates that the response of the system is linear. Note also that $\tilde{Q}$ can be written as $\tilde{Q} = N_b k_p^3 / n_0$, which is the ratio of the number of beam electrons to plasma electrons within a cubic plasma skin-depth. Thus we may also write the underdense condition, $n_b / n_0 = \tilde{Q}/(2\pi)^{3/2} k_p \sigma_z (k_p \sigma_r)^2$, and when $k_p \sigma_z$ is near unity, and (as is usual in experiment) $k_p \sigma_r << 1$ then $\tilde{Q} \approx 1$ implies that the beam is denser than the plasma. It should be noted in this regard that the experiments of Refs. 10 and 11 that are performed in the blowout regime (n_b / n_0 well in excess of unity) have beam-plasma systems yielding $\tilde{Q}$ values between 1.5 and 4.

Equation 9 has a temporal δ-function which we again integrate over, to obtain an inhomogenous modified Bessel equation in $\tilde{r}$

$$\frac{\partial^2 \mathbf{H}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \mathbf{H}}{\partial \tilde{r}} - \frac{\mathbf{H}}{\tilde{r}^2} - \mathbf{H} = \frac{\tilde{Q}}{\pi \tilde{a}^2}\delta(\tilde{r}-\tilde{a}), \qquad [11]$$

where $\mathbf{H} = \int_{\varepsilon-}^{\varepsilon+} \tilde{H}_\phi d\tau = \int_{\varepsilon-}^{\varepsilon+} \tilde{E}_r d\tau$. We interpret $\mathbf{H}$ as the total radial momentum impulse $\tilde{p}_r$, which in the non-relativistic limit is also approximately equal to $\tilde{J}_r$ immediately behind the beam. The solution to Eq. 11 is given by

$$\mathbf{H}(\tilde{r}) = \frac{\tilde{Q}}{\pi \tilde{a}}\begin{cases} K_1(\tilde{a}) I_1(\tilde{r}) & (\tilde{r} < \tilde{a}) \\ K_1(\tilde{r}) I_1(\tilde{a}) & (\tilde{r} > \tilde{a}), \end{cases} \qquad [12]$$

where I_1 and K_1 are modified Bessel functions.

We are interested in $\tilde{E}_z$ directly behind the beam, which is found by integrating Eq. 12

$$\tilde{E}_z(\tilde{r})\Big|_{\tau=\varepsilon+} = \int_{\infty}^{\tilde{r}} \mathbf{H}(\tilde{r}')d\tilde{r}' = \frac{\tilde{Q}}{\pi \tilde{a}^2}\begin{cases} 1-\tilde{a}K_1(\tilde{a}) I_0(\tilde{r}) \\ \tilde{a} I_1(\tilde{a}) K_0(\tilde{r}) \end{cases}. \qquad [13]$$

For $\tilde{a}$<<1, the field inside of the disk is nearly constant, and given by

$$\tilde{E}_z(\tilde{r})\Big|_{\tau=\varepsilon+} \cong \frac{\tilde{Q}}{\pi \tilde{a}^2}\left[1-\tilde{a}K_1(\tilde{a})\right] \cong \frac{\tilde{Q}}{2\pi}\left[\ln\left(\frac{2}{\tilde{a}}\right) - 0.577...\right], \qquad [14]$$

which is to leading order proportional to $\tilde{Q}/2\pi$. In physical units we may write Eq. 14 as

$$eE_z\Big|_{\tau=\varepsilon+} \cong 2e^2 k_p^2 N_b \ln\left(\frac{1.123}{k_p a}\right). \qquad [15]$$

Several comments arise from inspection of Eq. 15. The first is that the scaling of E_z with respect to wavenumber k is dominated by the factor k_p^2 that is typical of Cerenkov radiation [9], if we interpret k_p as the maximum allowable value of k radiated. The second comment is that the linear result is ill-behaved in the limit of $k_p a$<<1, as Eq. 15 predicts a logarithmic divergence in E_z. This pathology is a result of allowing $\tilde{J}_r$ (through $\mathbf{H}$) to diverge as r^{-1}. Previous analyses by Jackson [9] and Chen, *et al.,* [1] have attempted to mitigate this problem for the point charge limit by introducing a lower bound on r (or impact parameter b), and have chosen the Debye length[2]. This *ad hoc* way of removing the divergence of an ultra-

[2] It is not quite clear from Jackson's discussion if the Debye length is to be applied as the lower bound to b only in the case to non-relativistic particles (*cf.* Ex. 13.3, Ref. 9).

relativistic particle's energy loss in plasma has a dubious physical basis, however. Debye shielding places the scale of maximum distance that a particle's macroscopic field can be observed in the plasma, after a thermal equilibrium is established by the motion of the plasma electrons. However, here we are concerned with the minimum distance for which the fluid analysis is valid in describing the plasma electron response to an extremely fast transient, a particle with velocity much higher than the plasma electron thermal velocity.

IV. NONLINEAR PLASMA RESPONSE

As $\tilde{Q}$ is raised, we must consider the plasma electrons' relativistic response to large amplitude fields, under the general condition that $\tilde{H}_\phi = \tilde{E}_r$. It is most straightforward to evaluate this response in the rest frame of the beam [15], where the beam charge gives rise to only an electrostatic field. One can then find the radial momentum kick in this frame, and Lorentz transform back to the lab fram to obtain $\tilde{p}_r$ and $\tilde{p}_z$. We therefore find

$$\tilde{p}_r = \mathbf{H}, \qquad [16]$$

while the longitudinal momentum impulse is given by

$$\tilde{p}_z = \tfrac{1}{2}\tilde{p}_r^2. \qquad [17]$$

The passage of the beam induces not only an impulsive change in the radial momentum, but a longitudinal impulse in the positive (beam motion) direction.

For large $\mathbf{H}$, the plasma electrons experience a large forward momentum impulse, and can have relativistic v_z just after passage of the beam[3]. Equations 16 and 17 were verified by numerical integration in pulses of finite length; with their use the plasma electron's transverse velocity becomes

$$\tilde{v}_r = \frac{\mathbf{H}}{\sqrt{1+\mathbf{H}^2+\frac{1}{4}\mathbf{H}^4}} = \frac{\mathbf{H}}{1+\frac{1}{2}\mathbf{H}^2}. \qquad [18]$$

In order to relate this $\tilde{v}_r$ to $\tilde{J}_r$ we must multiply by $\tilde{n}$, which due to the change in $\tilde{v}_z$ directly after passage of the beam, is predicted with the aid of the continuity relation, Eq. 3, and Eq. 17 to be

$$\tilde{n} = (1-\tilde{v}_z)^{-1} = 1+\tfrac{1}{2}\mathbf{H}^2. \qquad [19]$$

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[3] If one chooses to derive Eqs. 15 and 16 in the laboratory frame, the solution of the "impulse" equations of motion must take into account a lengthened interaction time (a finite value evaluated in the d-function limit) as the plasma elections obtain longitudinal velocity, by multiplying by the factor $(1-v_z)^{-1}$.

Thus we are led to the remarkable result that the relativistically correct induced radial current is identical to the approximate, linear, non-relativistic expression,

$$\tilde{J}_r = \tilde{n}\tilde{v}_r = \left(1+\tfrac{1}{2}\mathbf{H}^2\right)\cdot\frac{\mathbf{H}}{1+\tfrac{1}{2}\mathbf{H}^2} = \mathbf{H}. \qquad [20]$$

Since the induced $\tilde{J}_r$ is unchanged from the linear case, the analysis of the decelerating field $\tilde{E}_z$ leading to Eq. 13 remains valid. Thus we see that the "linear" scaling observed in simulations of short pulse beam-excited wakefields may be understood partly on an analytical basis. The result in Eq. 20 arises from two effects which cancel each other: the induced $\tilde{v}_r$ saturates (at a value well below 1), yet the density enhancement due to longitudinal motion — a "snow-plowing" of the plasma electrons by the electromagnetic pressure — exactly makes up for this saturation, and the induced $\tilde{J}_r$ remains linear in $\tilde{Q}$. This snow-plowing is analogous to the scenario from laser wake-field acceleration, where the electromagnetic pressure in gradient of a short, intense laser gives rise to a density enhancement in the laser's leading edge.

In order for $\tilde{n}$ to be enhanced during beam passage, a net longitudinal displacement of the plasma electrons while the beam interaction must occur. Numerical integration of the equations of motion confirm this effect, which seems surprising in light of fact that no transverse displacement survives taking of the δ-function limit. This is as expected because the radial impulse is linear in the field strength ($\tilde{H}_\phi$, $\tilde{E}_r$), and the discontinuity in the radial velocity does not yield a displacement when integrated over a time interval inversely proportional to $\tilde{E}_r$ or $\tilde{H}_\phi$. Since the longitudinal impulse is *quadratic* in integrated field strength, however, a net positive displacement occurs. This effect can be alternatively be confirmed by performing an analysis in the rest frame of the beam and Lorentz transforming to the laboratory frame.

As a way of further investigating the nonlinear response of the plasma to the beam charge , we now examine the energy content of the excitation left in the beam's wake. The energy per unit length which must be supplied by the beam is found found by integrating the differential energy density in the plasma motion and field just behind the beam;

$$\begin{aligned}\frac{d\tilde{U}}{d\tilde{z}} &= 2\pi\int_0^\infty\left[\sqrt{1+\tilde{p}_r^2+\tilde{p}_z^2}-1\right]\tilde{n}\left(1-\tilde{v}_z\right)\tilde{r}\,d\tilde{r} + 2\pi\int_0^\infty \tfrac{1}{2}\tilde{E}_z^2(\tilde{r})\tilde{r}\,d\tilde{r} \\ &= \pi\left[\int_0^\infty \mathbf{H}^2\tilde{r}\,d\tilde{r} + \int_0^\infty \tilde{E}_z^2(\tilde{r})\tilde{r}\,d\tilde{r}\right].\end{aligned} \qquad [21]$$

Several comments are in order at this point: the first is that just behind the beam $\tilde{H}_\phi = \tilde{E}_r$=0, so there is no contribution to the field energy density from these field components. The second is that in order to find the differential spatial rate of mechanical energy deposition by the beam, one must take into account that the plasma electrons may be traveling in z, which means that the normalized mechanical energy density $\tilde{n}(\gamma-1)$ must be multiplied by $(1-\tilde{v}_z)$, as shown in in the first integral above. This factor, which is may be understood by analogy with a familiar effect found in the study of electromagnetic wake-fields in accelerators (although not as familiar as it perhaps should be to all, see the discussion in Ref. 15) removes the dependence of the energy loss on powers of $\tilde{Q}$ larger than 2. Evaluation of the integrals given above yields

$$\begin{aligned}\frac{d\tilde{U}}{d\tilde{z}} &\cong \frac{\tilde{Q}^2}{2\pi\tilde{a}^2}\int_{\tilde{a}}^{\infty}\left[K_0^2(\tilde{r})+K_1^2(\tilde{r})\right]\tilde{r}\,d\tilde{r}. \\ &= \frac{\tilde{Q}^2}{2\pi\tilde{a}^2}\left[1-\tilde{a}K_1(\tilde{a})I_0(\tilde{a})\right] \cong \tfrac{1}{2}\tilde{Q}\tilde{E}_z(\tilde{r})\Big|_{\tau=\varepsilon+}\end{aligned} \quad [22]$$

as expected. The factor of 1/2 is also familiar from the study of both plasma [2] and electromagnetic wake-fields — it arises from the averaging of the force over the bunch (zero at the front, maximum at the back), and taking the limit as the bunch length goes to zero.

As the results of Eqs. 16-20 concern beams of negligible length, they are applicable, in the limit $\tilde{a}\to 0$, to the case of a single particle. The effects of nonlinear plasma electron response do not, as might have been hoped, remove the logarithmic divergence seen in Eq. 14. Note that the logarithmic term in Eq. 15 corresponds to the familiar Coulomb logarithm [9], with an argument that is the ratio of the maximum to minimum impact parameter b, $\ln(b_{\max}/b_{\min})$. We deduce that the upper limit $b_{\max}=2/k_p$, while the lower limit in the analysis is a. The value of a in Eqs. 13-15 cannot be drawn towards zero without violating several assumptions of our analysis, however. The fluid assumption is fine; modeling the plasma electrons as a continuous fluid introduces errors not in the average energy loss, but in the fluctuations of this quantity. For ultra-relativistic particles, quantum mechanical effects constrain the minimum impact parameter [9] to $b_{\min}\cong\dfrac{\hbar}{m_e c}\sqrt{\dfrac{2}{\gamma}}$ through the uncertainty principle, however. Thus we write the energy loss rate for a point particle of charge q as

$$\frac{dU}{dz}\cong q^2k_p^2\ln\left(0.794\sqrt{\gamma}\,\frac{m_e c}{k_p\hbar}\right)\cong q^2k_p^2\ln\left(5\sqrt{\gamma}\,\frac{\lambda_p}{\lambda_c}\right), \quad [23]$$

where $\lambda_p=2\pi/k_p$ and λ_c are the plasma and Compton wavelengths, respectively. Note that both limits in the Coulomb logarithm can be viewed quantum mechanically, as the minimum quantum of energy loss (emission of a plasmon) in the plasma is in fact $\hbar\omega_p$, as has been verified experimentally for very thin foils [17].

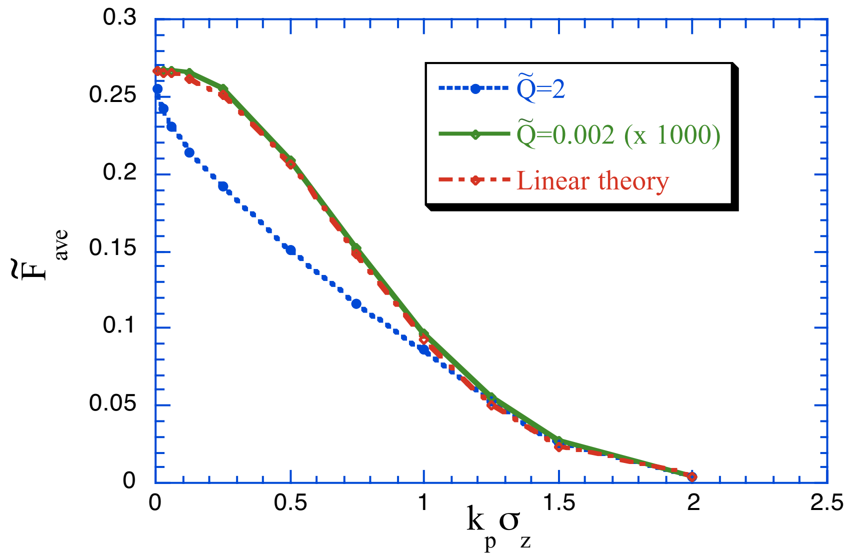

Figure 2. The average normalized energy loss rate of $\tilde{F}_{dec} = e\langle E_z \rangle / m_e c\omega_p$ of an electron beam with $k_p a = 0.2$, as a function of $k_p\sigma_z$, for $\tilde{Q}$=0.002 and (diamonds, solid line) 2 (circles, dotted line), from cylindrically symmetric fluid simulation, and linear theory (diamonds, dashed line).

V. SIMULATIONS: NONLINEAR RESPONSE AND FINITE BUNCH LENGTH EFFECTS

It is important to validate our infinitesimal length analysis, which by allowing an exact solution of the energy loss problem gives insight into the microscopic processes which are present in the nonlinear PWFA. Our analysis has been checked with numerical integrations of the fluid equations for finite length beams, having a longitudinal charge distribution, $\rho_b(z) \propto \exp(-z^2/2\sigma_z^2)$, and taking the limit that $k_p\sigma_z << 1$. In order to connect with the point beam limit, and to accurately quantify the energy imparted to the plasma, we compare the average on-axis beam energy loss rate, $(2\pi\sigma_z)^{-1}\int eE_z(z)\big|_{r=0}\exp(-z^2/2\sigma_z^2)dz$, for these cases with linear theory. The predictions of linear theory are obtained by using Eq. 14 to give the Green function (δ-function response), and performing a convolution integral [2] of the over the Gaussian pulse, to gives an average energy loss rate of $\dfrac{\tilde{Q}}{2\pi\tilde{a}^2}\left[1-\tilde{a}K_1(\tilde{a})\right]\exp\left(-k_p^2\sigma_z^2\right)$. The results of these simulations are shown in Fig. 1, which displays the average energy loss of a beam in the linear regime ($\tilde{Q}$=0.002), a comparison to linear analytical theory, and the nonlinear regime ($\tilde{Q}$=2). In the

$\tilde{Q}$=0.002 case, the fluid simulations agree extremely well with analytical predictions. For the case with $\tilde{Q}$=2, the simulations disagree with linear theory over a broad range of pulse lengths, but converge to the linear theory in the limit that $k_p\sigma_z <<1$, as expected from the conclusions we have drawn from Eqs. 16-20. Note that the numerical integration of the fluid equations is not easily stabilized when $\tilde{Q}$>2, and thus to perform further numerical investigations another tool must be adopted.

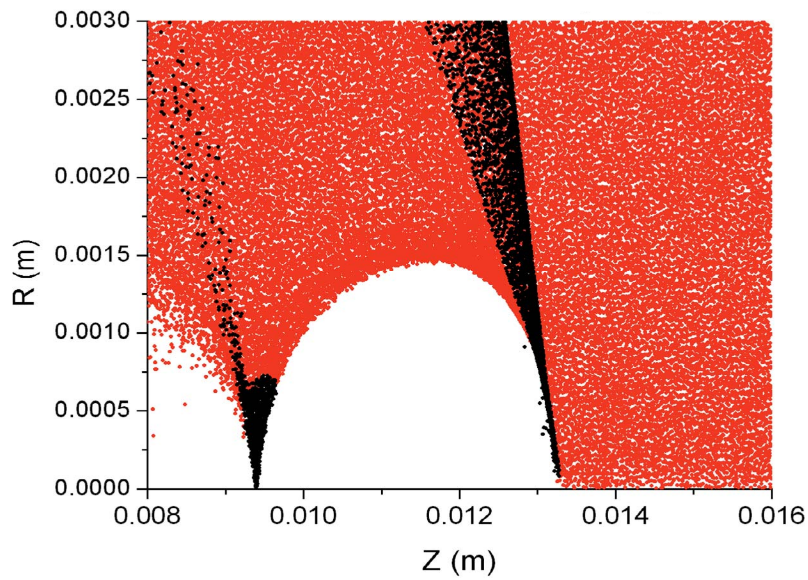

Figure 3. Configuration space from MAGIC cylindrically-symmetric PIC simulation with $\tilde{Q}\cong 20$, and $k_p\sigma_z = 0.11$, and $k_p a = 0.2$, and beam center at *z*=1.33 cm. Color code has black colored electron positions have *relativistic* positive momenta $p_z / m_e c > 1$, shaded (red, in color rendition) otherwise. The initially accelerated accelerated plasma electrons are just ahead of the blow-out region, where radial motion moves the electrons away from the beam channel.

In order to explore the nonlinear plasma response from the analytical $k_p\sigma_z <<1$ result, we have performed a series of simulations using a fully relativistic particle-in-cell code, MAGIC [18] and OOPIC [13]. Two codes were used to check consistency; both codes gave essentially the same answers for all comparisons. The first investigation undertaken using MAGIC concerned the validity of the physical model we have deduced from our analysis. In particular, one never expects a snowplow effect from linear theory. The two main characteristics of snow-plow are: 1) a forward velocity component, and 2) a plasma electron density increase, both in the region directly behind the beam. Both of these predictions are dramatically verified in Fig. 3, where we display a simulation with $\tilde{Q} \cong 20$, and $k_p\sigma_z = 0.11$, and $k_p a = 0.2$. Note that the "shock front" shown in this case, which consists of electrons moving both forward and radially outward at relativistic speeds is *not* a representation of the initial disturbance, which is localized around the longitudinal position of the beam. The front is canted because electrons in it that are further from the axis originated closer to the axis.

While the δ-function beam limit is relevant to verfication of the our theoretical analysis, and the point-charge case, it is not of highest practical interest in bunched beams, as it has often been argued that one should set $k_p\sigma_z \cong 1$ to optimize drive beam energy loss in a PWFA [2-6,10,11]. In order to explore the deviation in plasma response from the analytical $k_p\sigma_z << 1$ result, we have performed a series of PIC simulations. Taking the beam of radius $\tilde{a} = 0.2$, and Gaussian current profile with $k_p\sigma_z = 1.1$, we have scanned the charge from linear to very nonlinear response, $\tilde{Q} = 0.02$ to 200.

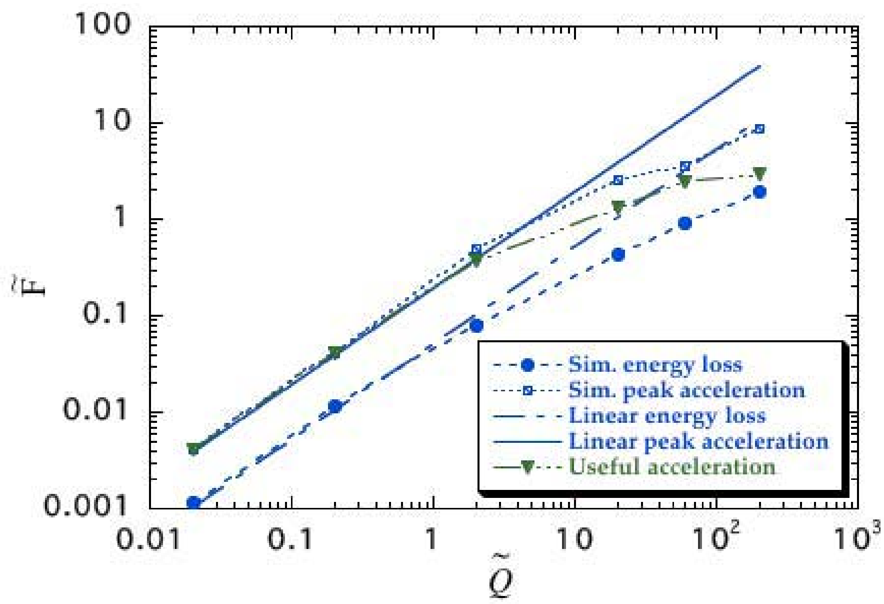

Figure 4. The average normalized energy loss rate of $\tilde{F}_{dec} = e|\langle E_z \rangle| / m_e c\omega_p$ of an electron beam with $k_p\sigma_z = 1.1$, $k_p a = 0.2$, as a function of $\tilde{Q}$, from linear theory (long-dash line) and self-consistent PIC simulation (circles, dashed line); the peak excited accelerating field behind the beam, $\tilde{F}_{\max} = e|E_{z,\max}| / m_e c\omega_p$, from linear theory (solid line) and PIC simulation (squares, dashed line); the useful field for acceleration (triangles, dash-dot line), defined by the geometry in Fig. 1.

The results of this parametric scan, as well as the analytical results of linear theory, are shown in Fig. 4. It can be seen that for the low amplitude cases $\tilde{Q} << 1$ that the linear theory predicts the average energy loss well. On the other hand, for $\tilde{Q} >> 1$, the energy loss is significantly smaller than that predicted by linear theory, by an order of magnitude at $\tilde{Q} = 200$. Note also that the energy loss rate does not grow appreciably as $\tilde{Q}$ is increased from 60 to 200. Similar results can be deduced from other published simulations [13]. From this behavior, it can be seen that the relativistic saturation of v_r and the snow-plow of the density for finite length beams do not cancel. The relative roles of these two effects, as well as that of changing plasma electron position during blow-out, can only be clarified by detailed simulation analysis.

We also plot the peak accelerating field excited behind the driving beam, and its predicted value from linear theory. The spike in the peak field that we have discussed above is magnified in the more nonlinear cases, causing a field *enhancement* relative to linear theory for $\tilde{Q} \cong 1$, and partly explaining why field saturation was not noted in previous simulation scans[4,8,12]. Even with this masking effect, however, the accelerating peak still displays saturation when $\tilde{Q} >> 1$. This saturation becomes severe for $\tilde{Q} > 100$.

In order to make illustrate the severity of nonlinear wave saturation effects for $\tilde{Q} >> 1$, we also plot the useful field for acceleration, as defined geometrically in Fig. 1. We can see that for $\tilde{Q} > 100$, the efficiency of exciting the acceleration field is very low. Experiments in this regime will be seriously impacted by nonlinear saturation. To quantify this, we replot in Fig. 5 the simulation data contained in Fig. 4, normalizing the results to the linear theory predictions. Additionally, we plot the approximate positions of existing (SLAC E162 and FNAL) and proposed (E164 and its upgrade) experiments on this normalized scale. It can be seen that even though present experiments are run in a nonlinear regime, they are minimally affected in all measures of field amplitude, due to the effects we have identified.

On the other hand, E164, in which the SLAC FFTB beam is compressed to 100 micron rms pulse length to obtain a $\tilde{Q} \cong 10$, displays a loss of a factor of over 2 in both deceleration of the drive beam and useful acceleration. This is already noted in E164 simulations [19], in which the expected scaling up of the gradient from lowering the bunch length by a factor of seven, or 49, is calculated to be below 30. There is now a proposed upgrade to this experiment, which we term E164*, where the beam is to be compressed another factor of 8, so with a correctly scaled plasma $\tilde{Q} \cong 80$. In this case we find that the peak accelerating field predicted from simulation is 22% of that given by linear theory. Even more impressive is the degradation of the useful field — it is only 7.7% of the linear scaled value. In real terms, this translates to an expected (from linear scaling) gradient of more than 600 GV/m being degraded to 50 GV/m.

One of the more interesting aspects of this study was the finding that the useful acceleration gradient decays much more rapidly at large $\tilde{Q} \cong 80$ than the average decelerating field. This can be reconciled with energy balance by noting that the much of the energy deposited in the plasma in the very large $\tilde{Q}$ case goes into driving plasma electrons to amplitudes where they do not return in one-half of an oscillation. This energy is lost from the wave, thus suppressing the achieved accelerating field behind the driving beam.

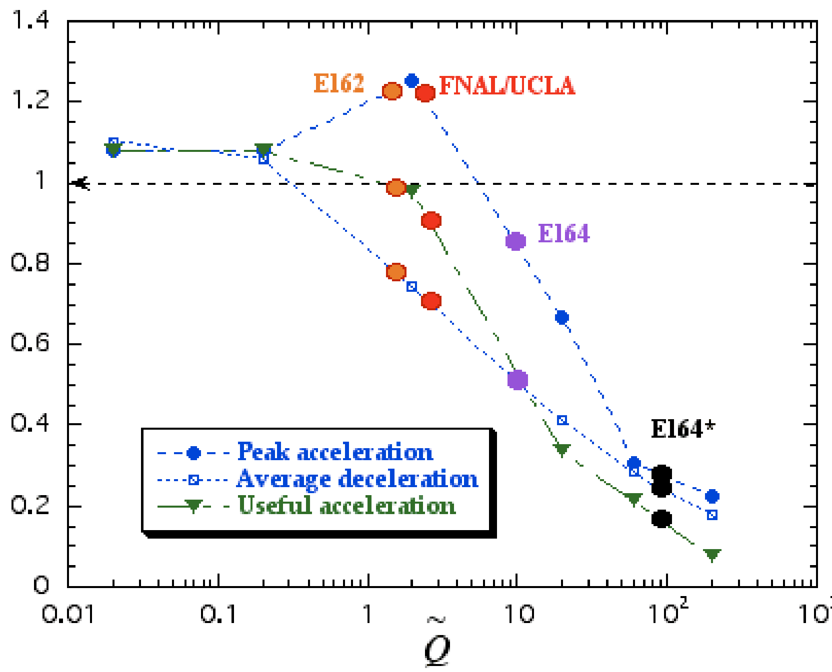

Figure 5. The average energy loss rate (squares, dotted line), peak accelerating field (circles, dashed line) and useful field for acceleration (triangles, dot-dash line), for the simulations shown in Fig. 4, normalized to the scaling predictions of linear theory. Present and future experiments are shown as circles on the simulation derived curves.

VI. CONCLUSIONS

In conclusion, we restate the most surprising of our results, that the fully relativistic response of a plasma to the passage of an ultra-short beam gives an induced electric field that is identical to the linear result. We have in the process identified a single parameter, the normalized charge $\tilde{Q}$, which identifies when a bunched beam is expected to give rise to nonlinear motion in the plasma. Further, this parameter may be used to predict the maximum physically achievable energy loss of a beam in plasma, Eq. 15. The interplay between the nonlinear effects —

relativistic saturation of transverse velocity, and snowplowing of plasma density — which cancel for infinitesimal, but not for finite, length beams must be studied in more detail by simulation. Such a study is now actively under way.

Even without this study, however, we may state that the previously proposed scaling of wakefield amplitudes as linear with k_p^2 (or σ_z^{-2} for constant $k_p\sigma_z$ and $k_p a$) is remarkably persistent until the normalized charge $\tilde{Q}$ well exceeds unity for finite length beams. On the other hand, when when $\tilde{Q}$ exceeds 10, coupling of the beam to the plasma becomes much less efficient, and field amplitudes to not grow as expected. It is further noted that because of this efficiency, it is unwise to adopt this regime of the PWFA. We thus suggest a practical limit of useful $\tilde{Q}$ to be 10.

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