NOISE-ENHANCED PARAMETRIC RESONANCE IN PERTURBED GALAXIES

IOANNIS V. SIDERIS

Department of Physics, Northern Illinois University, DeKalb, IL 60115

and

HENRY E. KANDRUP

Department of Astronomy, Department of Physics, and Institute for Fundamental Theory, University of Florida, Gainesville, FL 32611

ABSTRACT

This paper describes how parametric resonances associated with a galactic potential subjected to relatively low amplitude, strictly periodic time-dependent perturbations can be impacted by pseudo-random variations in the pulsation frequency, modeled as coloured noise. One aim thereby is to allow for the effects of a changing oscillation frequency as the density distribution associated with a galaxy evolves during violent relaxation. Another is to mimic the possible effects of internal substructures, satellite galaxies, and or a high density environment. The principal conclusion is that allowing for a variable frequency does not vitiate the effects of parametric resonance; and that, in at least some cases, such variations can increase the overall importance of parametric resonance associated with systematic pulsations.

Subject headings: galaxies: evolution – galaxies: kinematics and dynamics – galaxies: structure

1. Introduction and Motivation

A real galaxy is never a completely isolated time-independent equilibrium. To a lesser or greater extent it will always be subjected to a variety of perturbations. These include (nearly) regular perturbations, reflecting, *e.g.,* companion objects and/or (quasi-)normal modes triggered by close encounters with other galaxies (*e.g.,* Vesperini & Weinberg 2000) as well as more irregular perturbations, reflecting *e.g.,* internal substructures and/or larger numbers of nearby galaxies in a rich cluster. However, recent work (*e.g.,* Kandrup, Vass & Sideris 2003) has demonstrated that periodic and/or near-periodic perturbations can trigger parametric resonances which make a significant fraction of the stars execute chaotic orbits, leading to chaotic phase mixing and, oftentimes, readjustments in the bulk density distribution.

Most work hitherto on the effects of such parametric resonance has assumed a time-dependence characterised by a single fixed frequency. However, the frequency may not be exactly constant. Bulk changes in the density distribution during an epoch of violent relaxation will cause changes in the bulk potential and, hence, the natural pulsation frequency. And similarly, for a galaxy near equilibrium, even if most of the time-dependence is associated with a single normal mode with a well-defined frequency, one may have to allow for more irregular perturbations reflecting, *e.g.,* nearby galaxies or internal substructures.

In principle, these perturbations are deterministic. However, to the extent that they act in a 'near-random' fashion, one might wish to model them as coloured noise, *i.e.,* random 'kicks' of finite duration. As emphasised in Siopis & Kandrup (2000), any noisy process can be viewed as a superposition of periodic influences combined with random phases.

These considerations lead naturally to the issue considered in this paper, namely how parametric resonance associated with a single well-defined frequency can be impacted by perturbations inducing random fluctuations in the pulsation frequency. Do such fluctuations tend generically to enhance the effects of parametric resonance, or do they tend instead to weaken its effects? In particular, are there circumstances where even small random perturbations can dramatically increase the importance of parametric resonance and lead to potentially observable effects?

One principal conclusion is that a variable frequency need not vitiate the effects of parametric resonance; and, that, in some cases, such variations can make the effects of the resonance more pronounced. In particular, *even for comparatively low amplitude perturbations, 'fuzzing out' the frequency can lead to the displacement of appreciable quantities of stars and/or gas to larger radii and, in some cases, to their ejection from the galaxy.* This suggests that even comparatively weak interactions between galaxies (*e.g.,* the galactic 'harassment' of Moore, Lake & Katz [1998]) could contribute to the material often observed in intergalactic space in rich clusters, an effect normally attributed (*e.g.,* Bertin 2000) to galaxy-galaxy collisions and close encounters. This is an effect likely to be missed in N-body simulations, where the number of 'macroparticles' is small compared with the number of stars in a real galaxy.

2. What Was Done

As a simple example, one can consider orbits in a galaxy idealised as a spherically symmetric Dehnen (1993) model. Allowing for a time-dependent perturbation $m(t)$, suppose that, in appropriate units,

$$V(r,t) = -\frac{m(t)}{(2-\gamma)}\left[1-\left(\frac{r}{1+r}\right)^{2-\gamma}\right], \qquad (1)$$

with $r^2 = x^2 + y^2 + z^2$. Attention here will focus on $\gamma = 0$, $\gamma = 1$, and $\gamma = 3/2$, corresponding to galaxies with no cusp, a moderate cusp, and a steep cusp.

Now suppose further that, in the absence of perturbations, $m(t) \equiv 1$, and, in a first approximation, allow for a periodic time-dependent perturbation

$$m(t) = 1 + \delta m(t) = 1 + m_0 \sin \omega_0 t. \qquad (2)$$

For suitable choices of m_0 and ω_0, such a perturbation will trigger a parametric resonance which can make orbits chaotic and, in some albeit not all such chaotic orbits, induce significant changes in energy.

The objective then is to let the frequency to vary in a 'random' fashion which can be approximated as Gaussian coloured noise with a finite autocorrelation time. Assuming that $\omega(t) = \omega_0 + \delta\omega(t)$, with $\delta\omega$ sampling a stationary Ornstein-Uhlenbeck process (*e.g.,* Chandrasekhar 1943, Van Kampen 1981), the random process is characterised completely by the first two moments,

$$\langle \delta\omega(t) \rangle = 0$$

and

$$\langle \delta\omega(t_1)\delta\omega(t_2) \rangle = K(|t_1 - t_2|), \qquad (3)$$

where $\langle\ .\ \rangle$ denotes a statistical average and the autocorrelation function

$$K(|t_1 - t_2|) = \Delta^2 \exp(-|t_1 - t_2|/t_c) \qquad (4)$$

Here Δ represents the typical 'size' of the random component of ω and t_c the time scale over which it changes appreciably. Alternatively, perhaps more physically, the noise process can be characterised by t_c and a new quantity $\langle|\delta\omega|\rangle$, which corresponds to the mean value of $|\delta\omega(t)|$.

The specific focus here was to understand how such random frequency variations change representative orbits evolved in an otherwise strictly time-periodic potential. The first task, therefore, was to explore how initial conditions corresponding, in the absence of perturbations, to purely regular, *i.e.,* nonchaotic, orbits are impacted by an exactly periodic driving. This involved determining the extent to which the values of radius r and energy E were affected by the perturbation. In the absence of the driving, energy E is of course conserved. However, in the presence of driving, this is no longer true and, for appropriate choices of m_0 and ω_0, parametric resonance can induce large energy shifts.

Having determined the effects of such periodic driving, the second, and principal, task was to select 'interesting' values of unperturbed driving frequency ω_0 and explore how the dynamics is changed if ω is perturbed by allowing for random noises with different values of m_0, $\langle|\delta\omega|\rangle$, and t_c. Most of the experiments assumed $t_c = 80$, this corresponding to a time somewhat longer than a typical dynamical time, or orbital time scale, t_D (Merrritt & Fridman 1996). Thus, *e.g.,* for the two examples highlighted in this paper, $\gamma = 0$ and $E = -0.24$ corresponds to $t_D \sim 24$ and $t_c \sim 3.5t_D$, and $\gamma = 3/2$ and $E = -0.55$ corresponds to $t_D \sim 7$ and $t_c \sim 10t_D$. This choice is motivated by the expectation that relatively low amplitude pulsations could trigger changes in the density distribution of a galaxy, and hence the natural oscillation frequencies, on a time scale somewhat longer than the dynamical t_D but still short compared with the age of the Universe. A more systematic investigation of variable t_c will be presented elsewhere.

Since $\delta\omega$ is a random variable, a single integration of an initial condition does not suffice. Rather, for each choice of initial condition, m_0, ω_0, and $\langle|\delta\omega|\rangle$, 10000 different realisations were performed with the subsequent analysis focused on extracting statistical properties of the resulting orbits. The most important quantities to be extracted were the distributions of r_{max} and E_{fin}, the largest value of r and final value of E assumed by each of the noisy orbits, which provide information about the extent to which the noise-enhanced parametric resonance causes stars to be displaced to larger radii. (The quantity r_{max} would seem a better quantity to track then r_{fin} since, even if the energy increases monotonically, the radial coordinate will oscillate; and it is quite possible that, at the end of the integration, r is near a 'relative' perigalacticon.) Attention focused separately on initial conditions corresponding to (a) purely radial orbits and (b) orbits with a more isotropic distribution of velocities, earlier work (Terzić & Kandrup 2003) having shown that non-noisy parametric resonance can impact radial and nonradial orbits differently. (Especially for cuspy potentials, the Fourier spectra associated with radial orbits tend to have more power at higher frequencies, which allows for more efficient resonant couplings.)

This analysis is similar in spirit to recent work on noisy orbits in charged particle beams (Bohn & Sideris 2003), where the perturbations which one envisions reflect, *e.g.,* imperfections in the confining magnetic field and other details of an external environment. However, the work described here is different in at least two important respects. Firstly, the presence of a confining potential places an upper limit on the largest 'radius' that, in the absence of noise, a charged particle in the beam can achieve, which implies in turn that, for $\delta\omega = 0$, the response is a much smoother function of ω_0 than is the case for a galaxy. Secondly, and even more importantly, in the context of beam dynamics the principal issue of interest is in determining the largest values of r and E that any single beam particle can achieve, since the overriding concern is to prevent resonating particles from striking the walls of the accelerator. In the context of galactic dynamics, the actual distribution of maximum radii and energies would seem more important. Knowledge of these distributions can, *e.g.,* facilitate predictions about expected changes in the density distribution and/or estimates of the number of stars that might be ejected into intergalactic space.

3. What Was Found

Consider first orbits subjected to strictly periodic driving. In this case, for fixed amplitude m_0 and autocorrelation time t_c, the response to the time-dependence, as probed by r_{max} and E_{fin}, can ex-

hibit a comparatively sensitive dependence on the frequency ω_0. Because the response reflects a resonant coupling between (harmonics of) the driving frequency and the natural frequencies of the orbits, for sufficiently low and high ω_0 the driving has a comparatively minimal effect. In particular, the orbits are regular, *i.e.,* nonchaotic, as was the case for the unperturbed integrable model. For intermediate values, however, there are intervals where the orbits are strongly impacted by the periodic driving, which results, *e.g.,* in orbits acquiring large finite time Lyapunov exponents. An example of this behaviour for the $\gamma = 1$ Dehnen potential is exhibited in Fig. 1 of Terzic & Kandrup (2003), which also describes in detail the specific resonances that control the dynamics. Other examples for Plummer potentials are illustrated in Kandrup, Vass, & Sideris (2003).

For smaller amplitudes m_0, the dependence of quantities like r_{max} on ω_0 tends to be comparatively smooth, successive peaks in $r_{max}(\omega_0)$, corresponding to strongly chaotic orbits, being separated by fixed intervals. However, for higher amplitudes the response can be more complex, especially for orbits in cuspy potentials. As noted elsewhere (Terzić & Kandrup 2003), this is because Fourier spectra of cuspy orbits typically exhibit more structure than spectra of orbits in potentials with a smooth core.

Examples of this behaviour for orbits in pulsed Dehnen potentials with $\gamma = 0$ and $\gamma = 3/2$ are shown in Figure 1, which plots $r_{max}(\omega_0)$ for initial conditions evolved for a time $t = 512$. For each potential, results for two initial conditions are exhibited. Both correspond to an orbit with initial kinetic energy $T = -E/2$. However, in one case the initial velocity had components $v_x = v_y = v_z$; in the other the velocity was selected to yield a radial orbit. In each case, the initial $r = 0.25$, this corresponding to a star situated near the center of the galaxy. For the cuspless potential, the responses for these two initial conditions are quite similar, but for the cuspy potential the radial orbit was often ejected to a much larger radius. This is consistent with Terzic & Kandrup (2003), who found that, especially for cuspy potentials, radial orbits tended to be more susceptible to periodic driving.

Now allow the frequency to vary by including colored noise in the computations, supposing that the unperturbed ω_0 is in one of the resonant regions. At least for 'reasonable' choices of parameter values, *i.e.,* $\langle|\delta\omega|\rangle$ smaller than the width of the resonance, allowing the pulsation frequency to vary results on the average in orbits being displaced to larger radii. However, when $\langle|\delta\omega|\rangle$ is very small, such variations have a comparatively minimal effect. If $\langle|\delta\omega|\rangle/\omega_0$ is very small, say $< 10^{-4}$ or so, as a practical matter the frequency is 'nearly constant' in the sense that the variations typically have no appreciable impact on orbital structure, so that nothing really changes. Alternatively, when $\langle|\delta\omega|\rangle$ becomes too large these variations can suppress the transport of stars to larger radii and less negative energies. If the variation is too large, the effective frequency with which the orbit is perturbed can be shifted out of the 'interesting' resonant range, at which point the driving no longer has a significant effect. *The important point is that finite-sized variations that wiggle the frequency within the resonant range tend generically to increase the mean* r_{max}.

Examples of this behaviour are exhibited in Figure 2, which focuses on the same four initial conditions used to construct Figure 1, pulsed with frequency $\omega_0 = 0.6$ but allowing also for variations with variable $\langle|\delta\omega|\rangle$. In each case the data derive from 10000 noisy integrations, each proceeding for a time $t = 512$. The solid curve represents the largest radial excursion experienced by any of the 10000 orbits. The dotted curve represents the mean value of r_{max} averaged over the different orbits. For very small and large values of $\langle|\delta\omega|\rangle$, the frequency variations have only a comparatively minimal effect but, for $\langle|\delta\omega|\rangle \sim 10^{-3} - 10^{-1}$, the mean effect of this irregular time-dependence is substantially larger than for a purely periodic time-dependence. The range of values of $\langle|\delta\omega|\rangle$ resulting in a significant effect is comparable for radial and nonradial orbits. However, especially for the cuspy potential, within this range of values a varying frequency tends to have a larger effect on radial orbits.

As is illustrated in Figure 3, a varying frequency can also increase the largest E_{fin}. However, the mean value of E_{fin} does *not* tend to grow significantly.

Figure 2 demonstrates that the average effect of a variable frequency is to drive orbits to somewhat larger radii. However, it is also evident that some orbits can be expelled to *very* large values of r: If the orbit and the noise are appropriately 'tuned', the resonant coupling can have an enormous effect. For example, a radial initial condition in the $\gamma = 3/2$ Dehnen potential which, in the absence of a frequency variations, is restricted to radii $r < 1.2$ can, in the presence of such variations, be ejected to $r > 15$! The obvious question, however, is how rare such events actually

are. To address this issue, and to better understand the variations that a nonzero $\langle|\delta\omega|\rangle$ can trigger, it is useful to examine $f(r_{max})$, the distribution of values of r_{max} for the 10000 different noisy realisations.

Several examples of such distributions, generated for orbits in a $\gamma = 0$ Dehnen potential with $\omega_0 = 0.6$ and $m_0 = 0.05$, are presented in Figure 4. Perhaps the most obvious point here is that these distributions can be quite complex. In general they are asymmetric, and they often exhibit multiple peaks, suggestive of a superposition of seemingly distinct populations. For the lowest amplitude variations exhibited in that Figure, $\langle|\delta\omega|\rangle \approx 0.0002$, the mean values of r_{max} for the ensembles are very close to the r_{max} computed in the absence of any frequency variations, but the distributions $f(r_{max})$ still manifests significant structure. In particular, the distribution for the radial initial condition exhibits two peaks, the existence of which is of definite statistical significance. For the intermediate value $\langle|\delta\omega|\rangle \approx 0.06$, the shape of the distribution $f(r_{max})$ is relatively similar to the distributions in the upper panels. However, in this case most of the orbits have managed to reach larger radii; only a few are restricted to values smaller than the r_{max} associated with the driftless orbit. For the largest amplitude, $\langle|\delta\omega|\rangle \approx 0.25$, even though the mean change in r_{max} is relatively small, the distribution is quite broad; and it is clear that a majority of the orbits are actually restricted to smaller radii than was the driftless orbit.

These effects may perhaps be better understood by focusing on changes in energy rather than radius. This is illustrated in Figure 5, which exhibits $f(E_{fin})$, the distribution of final energies, computed for the same orbits used to generate Figure 4. Even visually, it is evident that, for the medium amplitude case, where the effect of the frequency variation is largest, $f(E_{fin})$ is reasonably well fit by a Gaussian with a mean equal to the E_{fin} associated with the zero variation orbit. It thus appears that, at least as viewed in energy space, the variations result in strictly random changes in energy superimposed on the energy shift that arises in the absence of the variation.

For the highest amplitude, the distributions are again Gaussian although the mean value is substantially smaller than the E_{fin} for the driftless orbit. This decrease can likely be attributed to the fact that, in this case, $\langle|\delta\omega|\rangle$ is sufficiently large that, in many cases, the effective frequency acting on the orbit is outside the resonant region, thus yielding a weaker response overall. (As is evident from Figure 1 a and b, the width of the resonance near $\omega = 0.6$ is $\lesssim 0.3$.)

For the lowest amplitude case, the situation is very different. For the isotropic initial condition, $f(E_{fin})$ has an extended low energy tail; for the radial case, it is bimodal, indicative of two distinct populations. Given that the initial $E = -0.24$, it is evident that the total time-dependence (almost) never extracts energy from the orbit. However, the lower energy portion of the distribution corresponds to orbits which experience only minimal changes.

The key to understanding this behaviour is the fact that chaotic orbits in a strictly periodically pulsed Dehnen potential divide empirically into two 'types', namely 'sticky' chaotic orbits which exhibit little if any systematic changes in energy and 'wildly' chaotic orbits which do exhibit systematic changes (Terzić & Kandrup 2003). Analysis of Fourier spectra indicates that the 'sticky' orbits are 'locked' to a harmonic of the driving frequency, which restricts the extent to which they can drift in energy, whereas the 'wildly' chaotic are unconstrained. However, this distinction is not absolute. For example, an orbit that starts as 'sticky' will eventually become unstuck and start drifting through energy space in much the same sense that, in a time-independent Hamiltonian system, a chaotic orbit originally trapped near a regular island can diffuse through cantori to move into the chaotic sea. This analogy with time-independent Hamiltonian systems suggests in turn that, just as noise can facilitate diffusion through cantori, corresponding to transitions between 'sticky' and 'wildly' chaotic behaviour (Pogorelov & Kandrup 1999), the introduction of weak noise into the driving frequency could facilitate transitions between orbits which do, and do not, exhibit systematic energy drifts.

If the frequency shifts are large, it might seem unlikely for 'stickiness' to persist for long time intervals since this would require the orbital frequencies to vary in just the right way as to remain locked to the driving. If, however, the frequency shifts are small, an orbit that started as sticky might be expected to remain sticky for a relatively long time; and similarly, if the orbit starts as wildly chaotic, there is the possibility that it will be 'kicked' into a frequency 'lock' and remain there for quite a while.

That such behaviour is possible is illustrated in Figure 6, which exhibits $E(t)$ and $r(t)$ for an orbit evolved in a pulsed $\gamma = 1$ Dehnen potential with $\omega_0 = 0.6$, $m_0 = 0.01$, and $\langle|\delta\omega|\rangle = 0.003$ for a total time $t = 2048$. Viewed over times $t < 500$ or so, the energy

does not appear to exhibit any systematic changes and the radial motion appears comparatively regular, even though an examination of the Fourier spectra suggests that the orbit is not regular. It is, however, evident that, at a slightly later time, the energy begins to evolve in a more irregular fashion and that, for $t > 1100$ or so, the energy increases systematically.

In any event, the behaviour exhibited in Figure 5 for the medium amplitude case becomes progressively more common for larger amplitudes and cuspier potentials, where the resonances become broader and 'sticky' chaotic orbits become less common.

Finally, as regards the genericity of the examples used to generate the Figures in this Section, two points should be made. The first is that, to a considerable extent, the value of the energy E is unimportant. In determining the response of an orbit what really matters is the overall complexity of the Fourier spectrum. A more complex spectrum tends to yield a more complex, and larger amplitude, response. What this *does* imply is that an orbit that is significantly impacted by a cusp will be more strongly effected by the periodic driving. This means that, for $\gamma \neq 0$, the response is larger for nearly radial orbits and/or very low energy orbits which spend a considerable amount of time in or near the cusp. Alternatively, for $\gamma = 0$ the potential reduces to a harmonic oscillator for $r \to 0$, so that, *e.g.,* very low energy orbits have especially simple spectra. The other point is that the exact value of the unperturbed driving frequency ω_0 is largely immaterial. The only really important thing is that ω_0 assume a value that triggers a resonant response. For very small amplitude m_0 this can require a bit of fine tuning; but, for larger m_0, where (see, *e.g.,* Kandrup, Vass, & Sideris 2003) the resonances are much broader, all that one really requires is that ω_0 be of order $1/t_D$. Varying the frequency within the resonant region has only minimal qualitative effects.

4. Conclusions and Interpretations

Periodic driving tends generically to pump energy into the stars in a galaxy, thus displacing them towards higher energies and larger radii, an effect resulting (*e.g.,* Kandrup, Vass & Sideris 2003, Terzić & Kandrup 2003) from a resonant coupling between the driving frequency and the frequencies of the orbits, which also make the orbits chaotic. Even relatively weak driving with fractional amplitude $\lesssim 0.05$ can have significant effects within a few tens of dynamical times; the effects of larger amplitude can be large enough to account for violent relaxation (at least in principle). Given, however, that this shuffling in energies involves a resonance, one might wonder to what degree this behaviour persists if the driving frequency is allowed to vary.

One principal conclusion of this paper is that such variations do not vitiate the effects of such resonant couplings. Indeed, allowing for modest variations in frequency tends on average to increase the maximum radii to which orbits are displaced. However, this does not imply that, on the average, such variations result in more energy being pumped into the orbits. To the extent that the orbits have all become 'wildly' chaotic, *i.e.,* that they have one or more positive Lyapunov exponents and that they are not 'locked' to the (near-constant) driving frequency, allowing for different random frequency variations leads to a Gaussian distribution of maximum energies E_{fin} centered about the maximum energy attained by an orbit subjected to strictly periodic driving. The systematic increase in the average r_{max} arises because realistic (near-)equilibria have phase space distributions which are monotonically decreasing functions of energy ($\partial f/\partial E < 0$), so that a symmetric spread in energies occasions an increase in the number of larger radii stars at the expense of stars at smaller radii.

More importantly, a random frequency can have a *very* large impact on at least a small number of stars. In particular, it is statistically probable that a few orbits will experience a noisy variable frequency which is 'just right' to give it a very large energy, thus displacing it to very large radii. (Alternatively, one might expect analogously that, for a fixed noisy variable frequency, there will be a small number of initial conditions resulting in 'just right' chaotic orbits which will be impacted significantly.) In particular, the computations described above show that, in some cases, even a perturbation of fractional amplitude as weak as $m_0 \sim 0.05$ can cause a star to reach a radius an order of magnitude larger than it would have reached if driven with a fixed frequency.

This is an effect likely to be missed in numerical simulations of galaxy evolution. As illustrated in Figure 7 for r_{max}, the distributions of values of r_{max} and E_{fin} decay exponentially for large values. This implies, *e.g.,* that the largest value of r_{max} computed for a collection of N orbits should increase logarithmically in N, an effect illustrated in Figure 8. Given, however, that few simulations of galaxies in-

corporate more than $\sim 10^7$ macroparticles, it would seem highly probable that such simulations could miss rare 'events' that arise at least occasionally in systems with $> 10^{11}$ stars.

The computations described here suggest strongly that subjecting a galaxy to time-dependent perturbations which can be idealised as nearly periodic oscillations with variable frequency can have significant, potentially observable, effects. However, there remain a large number of lacuna, some of which are currently under investigation (Kandrup, Sideris & Bohn 2003). Most obvious, perhaps, is the question of how the results derived here depend on the autocorrelation time t_c. There are indications (Kandrup, Vass & Sideris 2003) that the precise value of t_c may not be all that important, but this remains to be checked.

Also important is the issue of how coloured noise can facilitate transitions between 'sticky' and 'wildly' chaotic orbits. Earlier work (Terzic & Kandrup 2003) has shown that chaotic orbits subjected to purely periodic driving divide into two 'types', separated by 'entropy barriers', namely 'sticky' chaotic orbits which remain in the same phase space region without exhibiting systematic secular variations in energy and 'wildly' chaotic orbits where changes in energy typically grow diffusively. The obvious question, then, is to what extent the introduction of noise might accelerate transitions between orbit types. One knows, for example, that, in a time-independent Hamiltonian system, noise can facilitate diffusion through cantori or along the Arnold web (Lichtenberg & Wood 1989, Pogorelov & Kandrup 1999).

Practical applications also require that the results described here be generalised to allow for unperturbed potentials which are nonspherical, especially those admitting chaotic orbits. Earlier work (Kandrup, Sideris, Terzić & Bohn 2003) has shown that time-independent potentials that already admit chaos can be significantly more susceptible to time-periodic perturbations, especially those characterised by relatively low frequencies (as might be appropriate for modeling an external environment); and it would not seem unreasonable to conjecture that frequency variations might also prove more important in such settings.

Most importantly, however, it would seem imperative to consider the statistical effects of a noisy frequency on ensembles of initial conditions that constitute fair phase space samplings, so as to facilitate detailed predictions about the macroscopic properties of 'real' galaxies. In the first instance this could be done by considering *e.g.,* microcanonical samplings of constant energy hypersurfaces in fixed potentials. Ultimately, however, this problem can – and will – be addressed in the context of fully self-consistent simulations.

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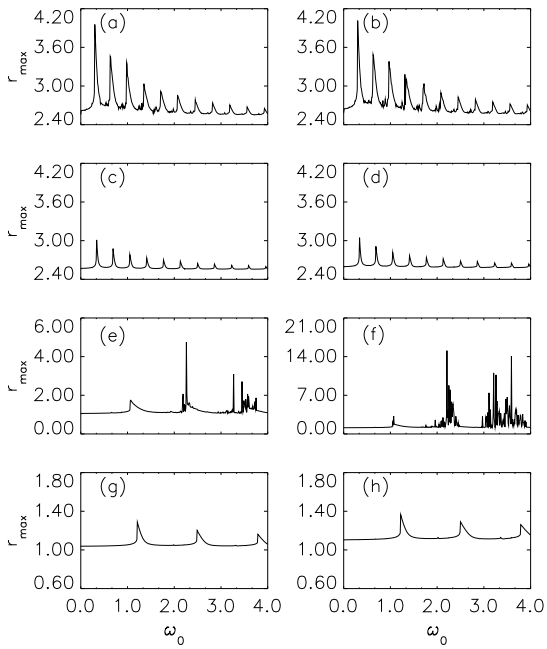

Fig. 1.— (a) r_{max}, the maximum value assumed by the radial coordinate r for orbits generated from an isotropic initial condition with $E = -0.24$, evolved deterministically in a pulsed $\gamma = 0$ Dehnen potential with $m_0 = 0.05$ and variable frequency ω_0. (b) The same for a radial initial condition with $E = -0.24$. (c) The same isotropic initial condition evolved with $m_0 = 0.01$. (d) The same radial initial condition evolved with $m = 0.01$. In each case frequency was sampled at intervals $\delta\omega_0 = 0.01$. (e) - (h) The same as the preceding, now allowing for initial conditions with $E = -0.55$ evolved in a $\gamma = 3/2$ Dehnen potential.

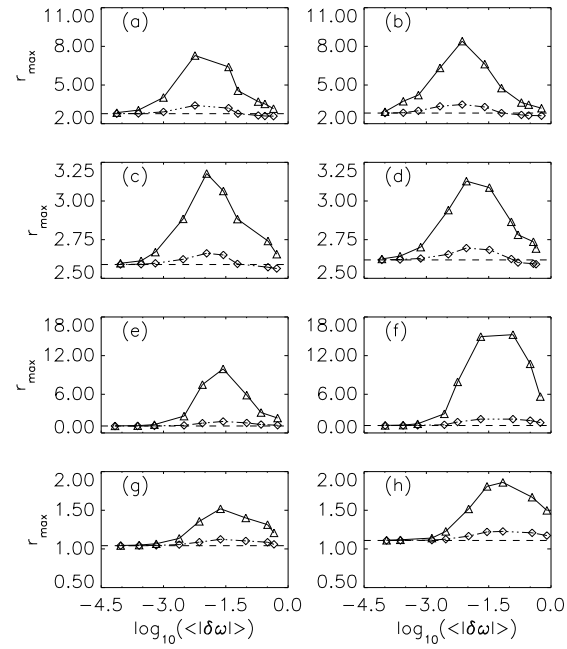

Fig. 2.— (a) The largest value (solid) and the mean value (dotted) of the maximum radial coordinate r_{max} for noisy orbits generated from an isotropic initial condition with $E = -0.24$, evolved in a $\gamma = 0$ Dehnen potential with $\omega_0 = 0.6$, $m_0 = 0.05$, and $t_c = 80$. (b) The same for a radial initial condition with $E = -0.24$. (c) The same isotropic initial condition evolved with $m_0 = 0.01$. (d) The same radial initial condition evolved with $m = 0.01$. The dashed line corresponds to a deterministic orbit with $\delta\omega \equiv 0$. (e) - (h) The same as the preceding, now allowing for initial conditions with $E = -0.55$ evolved in a $\gamma = 3/2$ Dehnen potential.

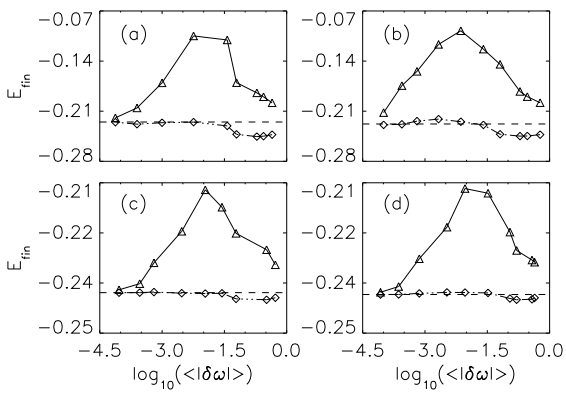

Fig. 3.— The analogue of Figure 2 (a) - (d), now exhibiting the maximum energy E_{fin} rather than r_{max}.

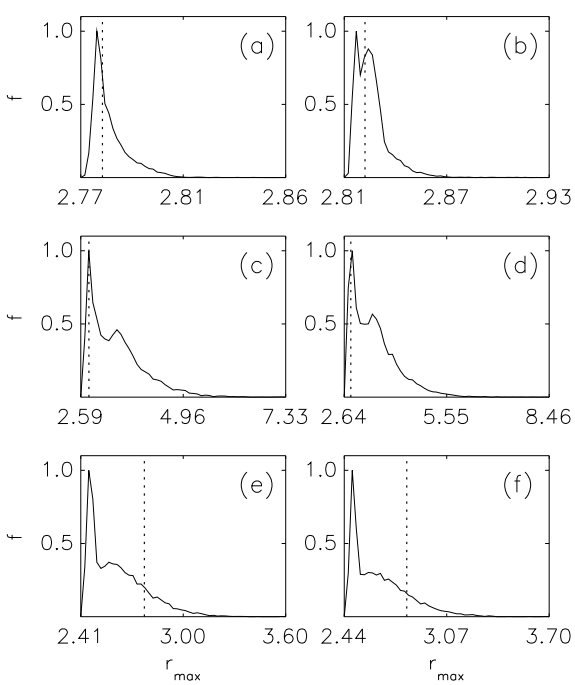

Fig. 4.— The distribution of maximal radial excursions, r_{max} for the two initial conditions of Fig. 1 evolved in the $\gamma = 0$ Dehnen potential with $\omega_0 = 0.6$ and $m_0 = 0.05$. The left panels are for the isotropic initial condition with (top to bottom) $\langle|\delta\omega|\rangle \approx 0.0002$, 0.006, and 0.25. The right panels are for the radial initial condition with the same values of $\langle|\delta\omega|\rangle$. The dotted vertical line exhibits r_{max} for the same initial condition subjected to purely periodic driving.

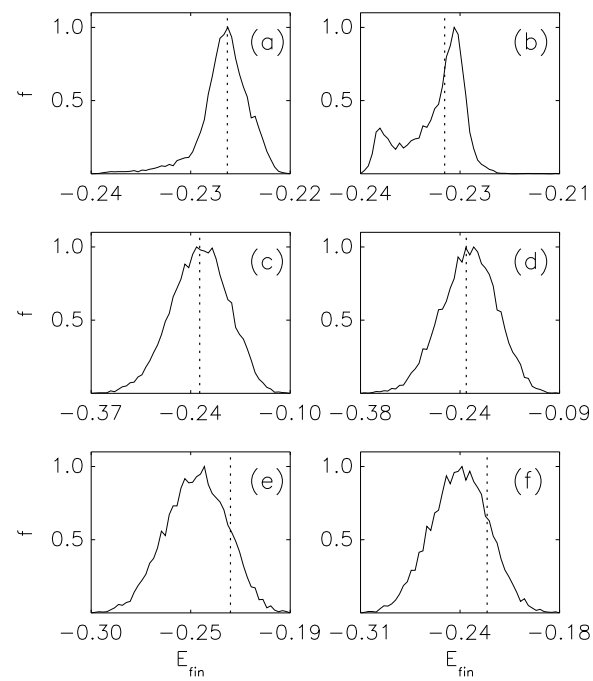

Fig. 5.— The distribution of final energies E_{fin} for the same data analysed in the preceding Figure. The dotted vertical line represents E_{fin} for the same initial condition subjected to purely periodic driving.

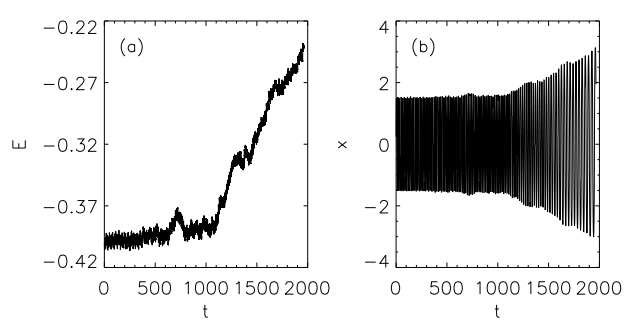

Fig. 6.— (a) The energy $E(t)$ of a radial orbit with $E(0) = -0.4$ and $y \equiv z \equiv 0$ evolved in a pulsed $\gamma = 1$ Dehnen potential with $\omega_0 = 0.6$, $m_0 = 0.01$, and $\langle|\delta\omega|\rangle = 0.003$. (b) The coordinate $x(t)$ for the same orbit.

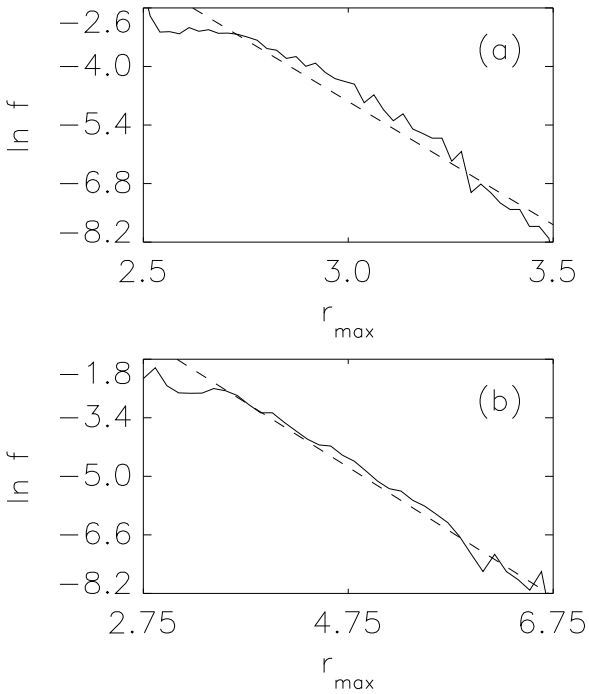

Fig. 7.— (a) Exponential fall-off in the distribution $f(r_{max})$ for an isotropic initial condition evolved in a pulsed $\gamma = 0$ Dehnen potential with $\omega = 0.6$, $m_0 = 0.05$, and $t_c = 80$, and $\langle|\delta\omega|\rangle = 0.2$. (b) The same for $\langle|\delta\omega|\rangle = 0.007$.

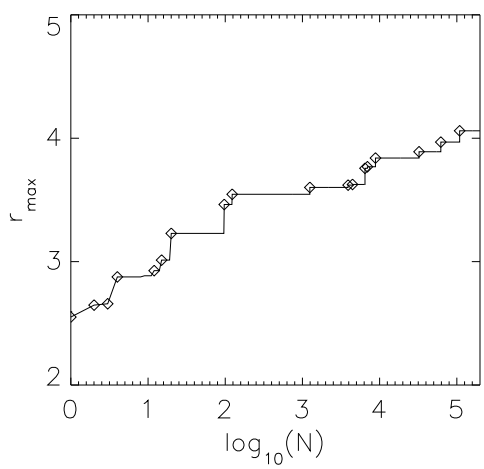

Fig. 8.— The largest value of r_{max} for a radial initial condition evolved in a pulsed $\gamma = 0$ Dehnen potential with $\omega = 2.8$, $m = 0.05$, $t_c = 80$, and $\langle|\delta\omega|\rangle = 0.08$, plotted as a function of the number N of noisy realisations.