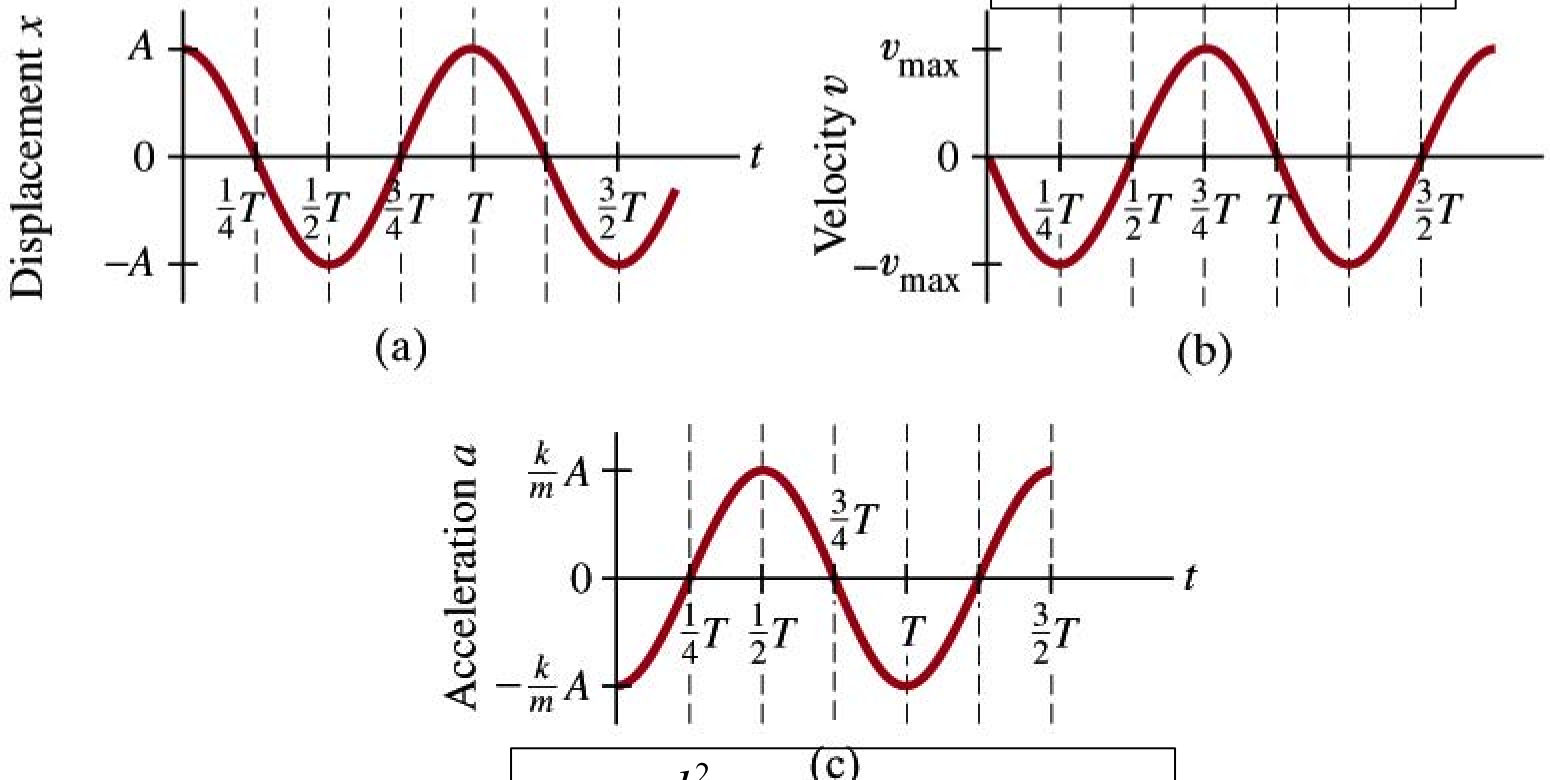
Status: Unit 4 - Circular Motion and Gravity

- ✓ Uniform circular motion, angular variables, and the equations of motion for angular motion (3-9, 10-1, 10-2)
- ✓ Applications of Newton's Laws to circular motion (5-2, 5-3, 5-4)
- ✓ Harmonic Motion (14-1, 14-2)
- The Universal Law of Gravitation, Satellites, and Kepler's Law's (Chapter 6)

$$x(t) = A\cos(\omega t + \phi)$$
$$x_{max} = A$$
$$x_0 = x(0) = A\cos(\phi)$$

$$v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$
$$v_{max} = \omega A = \sqrt{\frac{k}{m}}A$$
$$v_0 = v(0) = -v_{max}\sin(\phi)$$



$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$
$$a_{max} = \omega^2 A = \frac{k}{m}A$$
$$a_0 = a(0) = -a_{max}\cos(\phi)$$

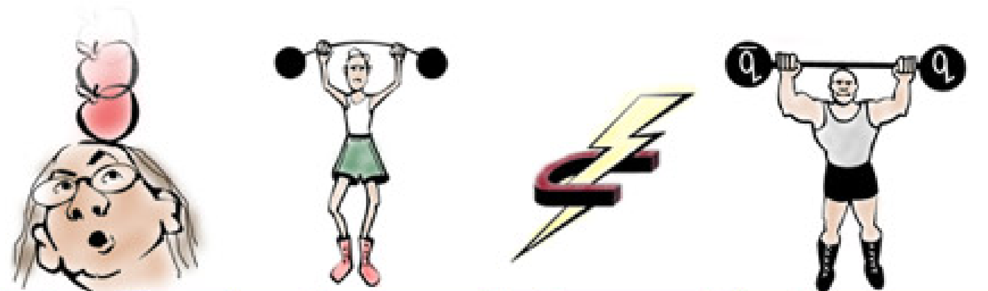
The Universal Law of Gravitation

- Explained the motion of the planets and acceleration of gravity.
- Along with the three laws of motion, one of Newton's greatest contribution to science.
- Published in the Principia (1687).
- Led to a fully deterministic description of the mechanical universe.
- The deterministic nature not "overturned" until the quantum revolution of the 20th century.

- Let's try to follow Newton's logic
  - Sine $\Sigma F=ma$ there must be a force associated with the acceleration objects feel <u>anywhere</u> on the surface of the earth.
  - The ubiquitous nature of this acceleration led Newton to conclude that the Earth itself exerts the gravitational force.
  - The observation that gravity acts at the surface, at the tops of trees and on mountains suggests the force also acts on the moon! (Here is where you can insert the apocryphal story of the falling apple…)

Action at a Distance?

- Newton's logic led to the gravitational force "acting at a distance".
- Common sense seems to suggest that contact is require for forces to exert themselves. This is true of friction, the normal force, tension…
- This was hard to accept at the time and with the quantum revolution a solution was at hand…
- And it's still not fully resolved!



	Gravity	Weak (Electroweak)	Electromagnetic (Electroweak)	Strong
Carried By	Graviton (not yet observed)	W^+ W^- Z^0	Photon	Gluon
Acts on	All	Quarks and Leptons	Quarks and Charged Leptons and W^+ W^-	Quarks and Gluons

The Form of the Universal Law of Gravitation

- On the surface we know empirically that the acceleration of gravity is g= $9.80 m/s^2$.
- Since the moon moves in circular orbit we can us $a=v^2/r$ to calculate $a_R \sim g/3600$.
- But the moon is 60 earth radii from the center of the earth.
- These last two observations suggest that the force of gravity between two objects is inversely proportional to the distance squared or

$$\text{Force of Gravity} \propto \frac{1}{r^2}$$

- And the 2^{nd} law clearly states that at the Earth's surface the force of gravity is proportional to the mass of the falling body, generalizing this to any distance we have,

$$\text{Force of Gravity} \propto \frac{m_B}{r^2}$$

- The third law requires that each force have an equal and opposite force. This suggests that the earth feels a force as well and by symmetry proportional to the earth's mass:

$$\text{Force of Gravity} \propto \frac{m_E m_B}{r^2}$$

- Newton also observed that the force holding the planets in their orbits varies with the inverse square of their distance to the sun and another manifestation of the gravitational force.
- This naturally leads to the idea that all objects exert such a force on one another.

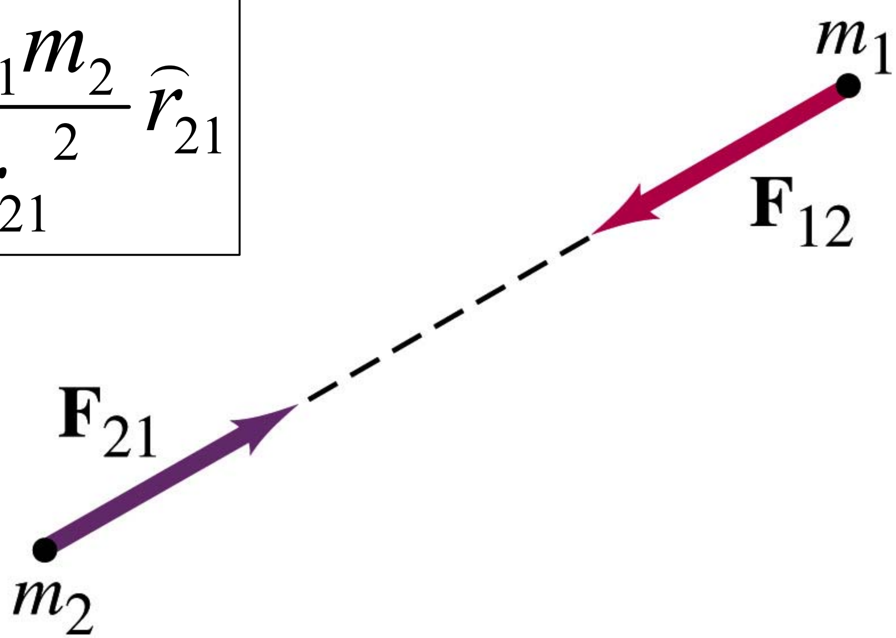
- The law of universal gravitation: Every pair of objects in the universe attract each other with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. The forces act along the line joining the two objects.

$$F = \frac{G m_1 m_2}{r^2}$$

- The constant of proportionality is experimentally determined but is the same for all objects.

The Vector Form of Universal Gravitation

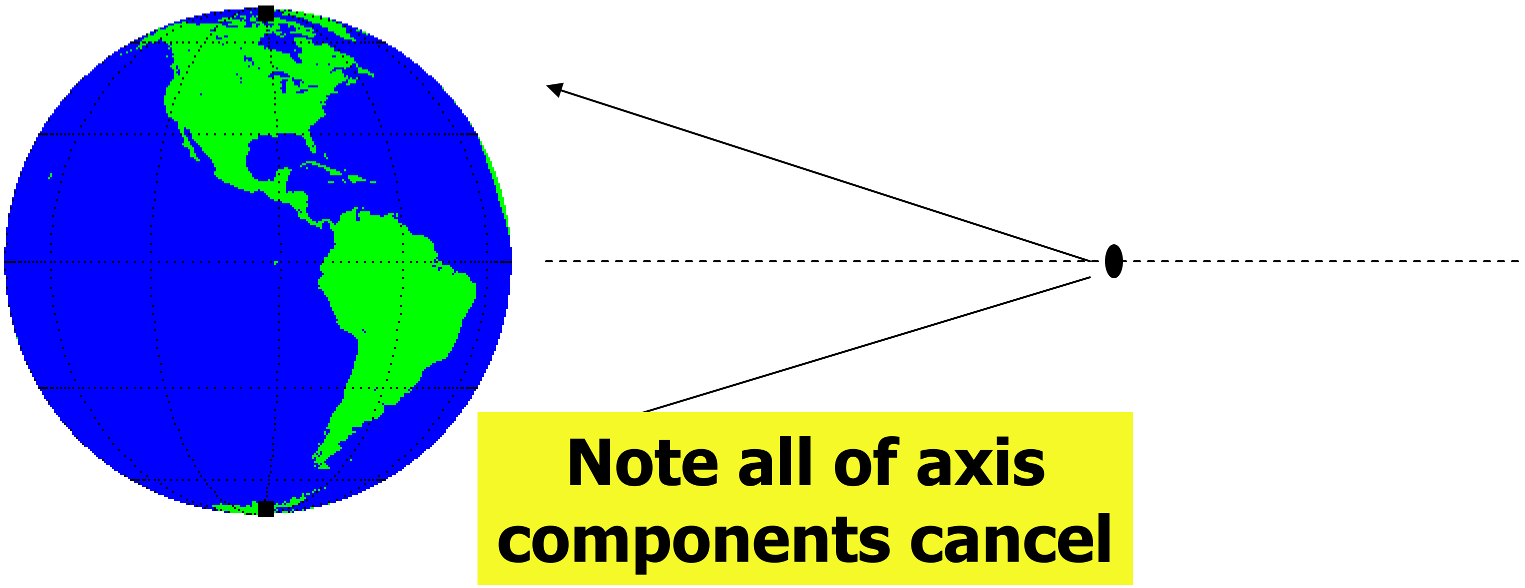
$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{21}^{\ 2}}\hat{r}_{21}$$



$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \ldots = \sum_{i=2}^{n} F_{1i}$$

Using the calculus and symmetry Newton showed that for a spherical object the force of gravity "emanates" from the center.

See Appendix C for a rigorous derivation.



On the Surface of Earth

- Applying the equation of Universal Gravitation to an object of mass m on the surface of the earth
  - $m_1 = m_E$
  - $m_2 = m$
  - $r = r_E$
- But from our earlier observations and lessons we also know that the force is equal to mg

$$F = G\frac{m_1 m_2}{r^2} = G\frac{m_E m}{r_E^2}$$

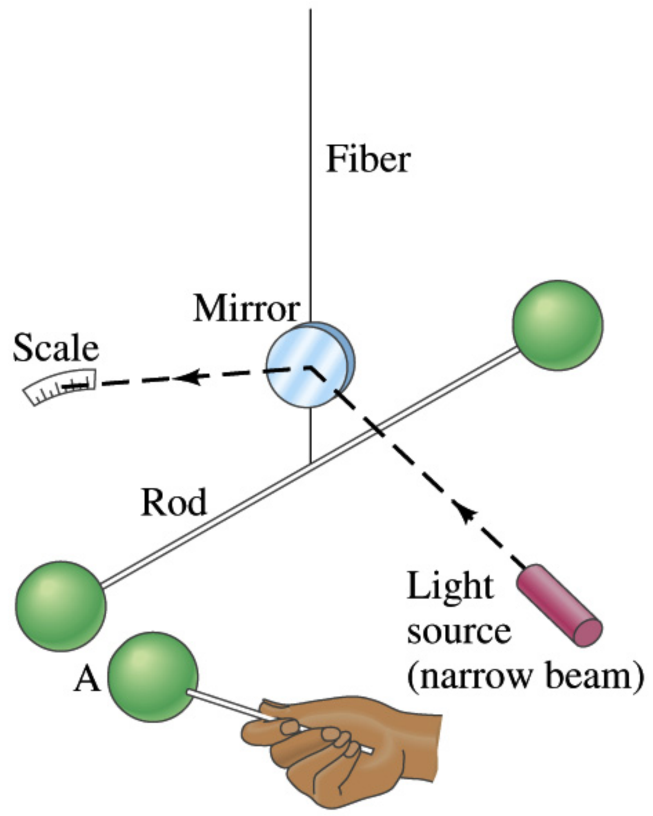
$$mg = G\frac{m_E m}{r_E^2} \rightarrow$$

$$\boxed{g = G\frac{m_E}{r_E^2}}$$

- Thus g is determined by the universal constant and the mass and size of the earth.

Determining the Universal Constant, G

- Must be small because we don't feel it in our everyday experiences.
- First measured by Henry Cavendish using a torsion balance.
- $G=6.67 \times 10^{-11}\ Nm^2/kg^2$



A chemist and physicist - he was really after the mass of the earth (to 1%)

Recall we derived that $g = G\frac{m_E}{r_E^2}$

solving for m_E

$$m_E = \frac{g r_E^{\ 2}}{G} = \frac{(9.80 m/s^2)(6.38\times 10^6 m)^2}{6.67\times 10^{-11} Nm^2/kg^2} =$$

$$m_E = 5.98\times 10^{24} kg$$

Magnitudes

- What's the gravitational force between
  - two 100 kg individuals sitting on a bench 0.5 meter apart?
  - the earth and a 100kg person?
  - the earth and the moon?
  - the sun and the moon?

On the bench :

$$F = \frac{Gm_1m_2}{r^2}$$

$$F = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(100kg)(100kg)}{(0.5m)^2}$$

$$F = 2.7\times10^{-6}\,N$$

The earth and a person : $F = mg = 980N$

Earth and moon :

$$F = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(7.35\times10^{22}\,kg)(5.98\times10^{24}\,kg)}{(3.84\times10^{8}\,m)^2}$$

$$F = 1.99\times10^{20}\,N$$

Sum and moon :

$$F = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(7.35\times10^{22}\,kg)(1.99\times10^{30}\,kg)}{(1.50\times10^{11}\,m)^2}$$

$$F = 4.34\times10^{20}\,N$$

The change on top Mt. Everest

- Estimate the effective value of g at the peak of Mt. Everest, 8,848m above sea-level.

Starting with $g = G\frac{m_E}{r_E^2}$ we use the larger radius

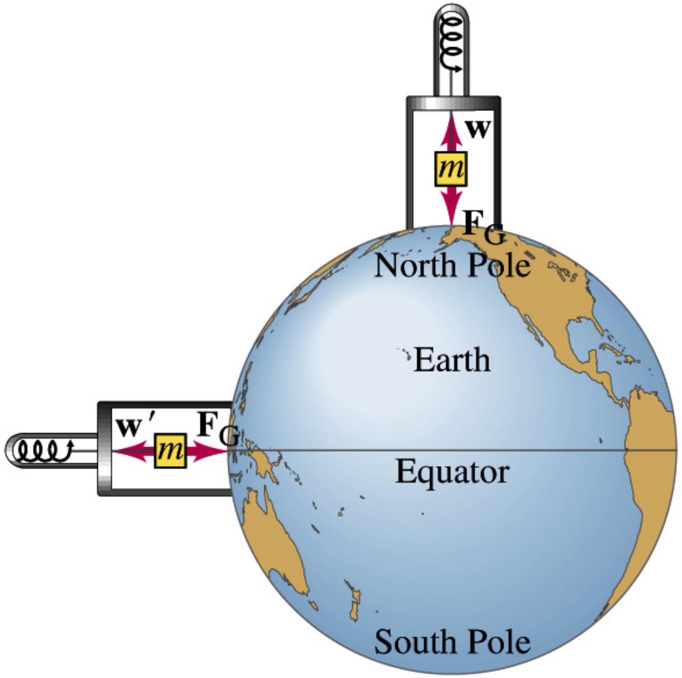
$$g' = 6.67\times10^{-11}\,Nm^2/kg^2\,\frac{5.98\times10^{24}\,kg}{(6{,}380{,}000m + 8800m)^2}$$

$$= 9.77m/s^2$$

- Which is about 0.3% lower than at sea-level. Something easily measurable with a sensitive scale.

The Effect of the Earth's Rotation on g

- Because of centripetal acceleration g is a bit different at the pole and the equator.
- Consider the two freebody diagrams at the right showing a mass hanging from a spring scale.
- At the north pole the 2nd law gives: $mg-w=0$ and $w=mg$.
- However at the equator there is also centripetal acceleration as the mass rotates around the center of the earth. The 2nd law gives: $mg-w'=mv^2/r_E$.
- This can be reorganized to $w'=m(g-v^2/r_E)$



- Which can be interpreted as an effective $g' = g - v^2/r_E$
- The change in the acceleration of gravity is thus just v^2/r_E.
- The velocity is given simply by $v = 2\pi r_E/T$, substituting we find $\Delta g = g - g' = 4\pi^2 r_E/T^2$
- And using
  - $r_E = 6.38 \times 10^6 m$
  - $T = 1 day = 8.64 \times 10^4 s$

- Plug and chug gives: $\Delta g = 4\pi^2 (6.38 \times 10^6 m)/(8.64 \times 10^4 s)^2 = 0.0034\ m/s^2$.
- Note:
  - This is about a 0.03% reduction and an order of magnitude less that the effect on Mt. Everest.
  - Because the centripetal acceleration depends on the distance from the axis of rotation it varies with latitude. This also means the acceleration vector does not point directly downward except at the equator and pole!

Satellites in Circular Orbits

- Now for a bit of fun!
- With our tools & concepts we can study some of the characteristics of satellites.
- If an object is to stay in circular orbit around the earth a centripetal force must continually turn the velocity vector, gravity!
- Using the 2nd Law and setting the mass of satellite to m, we find a precise relationship between the velocity and the radius of a satellite's orbit:

$$F = \frac{Gmm_E}{r^2} = m\frac{v^2}{r}$$

Note the mass of the satellite cancels:

$$\frac{Gm_E}{r^2} = \frac{v^2}{r} \text{ as does one power of r:}$$

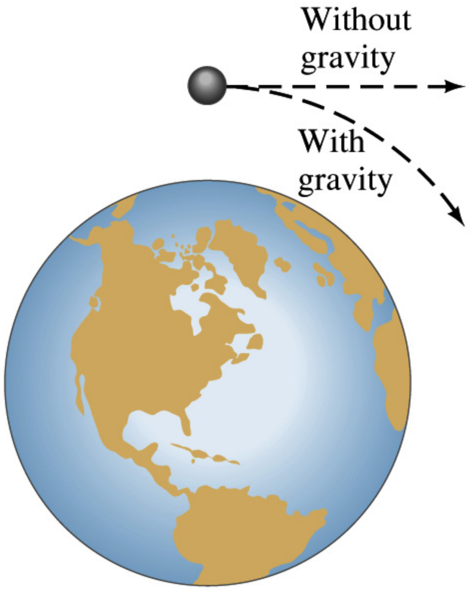
$$\frac{Gm_E}{r} = v^2 \text{ solving for v we see:}$$

$$v = \sqrt{\frac{Gm_E}{r}}$$

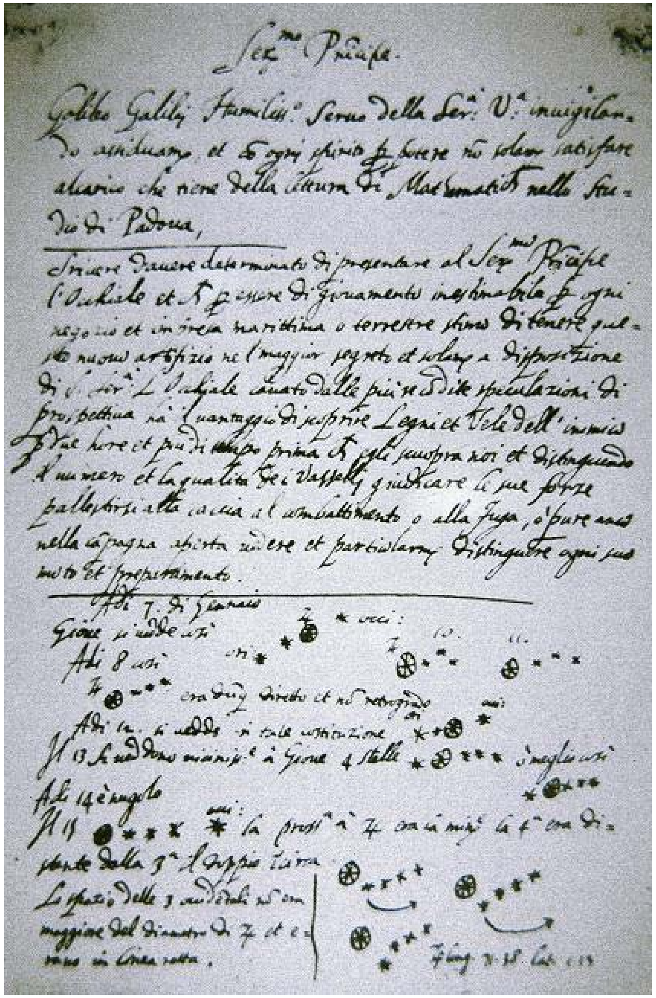
- One speed for a fixed radius.
- v inv. prop to 1/sqrt(r).
- v same for any mass

The Essence of Orbiting Satellites

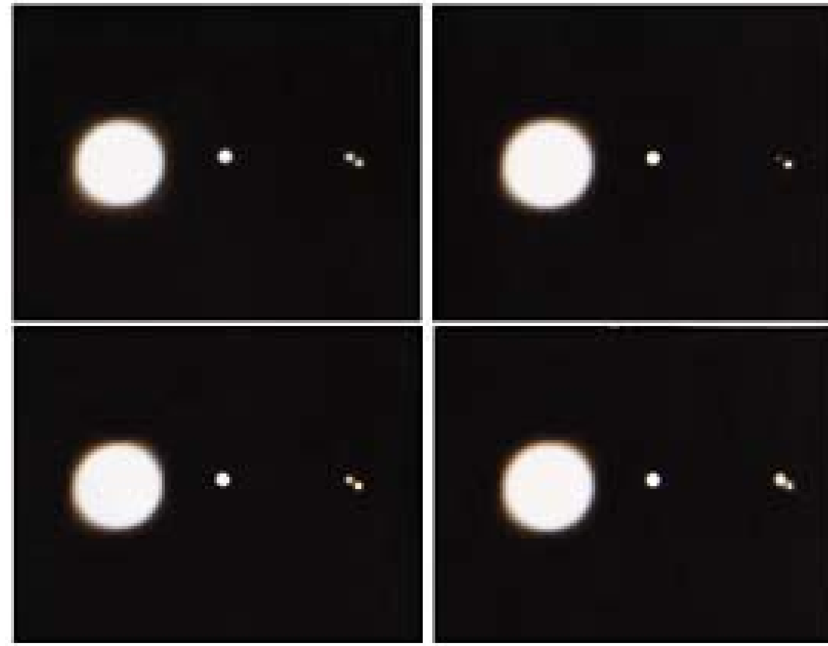
- The tangential velocity is what keeps the object in orbit.
- The centripetal acceleration provided by gravity merely turns this velocity into the uniform circular path.
- The relationship between velocity, G, the mass of the earth and r gives us the precise interplay between the kinetic and geometric variables.
- The result applies to any astronomical system. Just replace m_E with the mass about which an object is orbiting.
- We could use it to study the motion of the planets around the sun if we use the solar mass or of satellites about Jupiter if we use Jupiter's mass.



$$v = \sqrt{\frac{Gm_E}{r}}$$



Galileo Galilei first observed the moons of Jupiter on January 7, 1610 through a homemade telescope



Jupiter and its moons (from left to right) Ganymede, Europa being eclipsed, and Callisto.

Example: Hubble Telescope

- Determine the speed of the Hubble space telescope orbiting at a height of 596 km above the earth's surface.
- To start we just need to recognize that the radius of the orbit is equal to the radius of the earth plus the height above the surface and apply our new found equation.

$$r = r_E + h = 6.38\times10^6 m + 0.596\times10^6 m$$

$$r = 6.98\times10^6 m, \text{ thus}$$

$$v = \sqrt{\frac{Gm_E}{r}} = \sqrt{\frac{(6.67\times10^{-11} Nm^2/kg^2)(5.98\times10^{24} kg)}{6.98\times10^6 m}}$$

$$v = 7.56\times10^3 \sqrt{\frac{(kg - m/s^2)(m^2/kg^2)(kg)}{m}}$$

$$v = 7.56\times10^3 \sqrt{\frac{(m/s^2)(m^2)}{m}} = 7.56\times10^3\sqrt{m^2/s^2}$$

$$v = 7.56\times10^3 m/s = 17{,}000 mph$$

Example: A Distant Galaxy

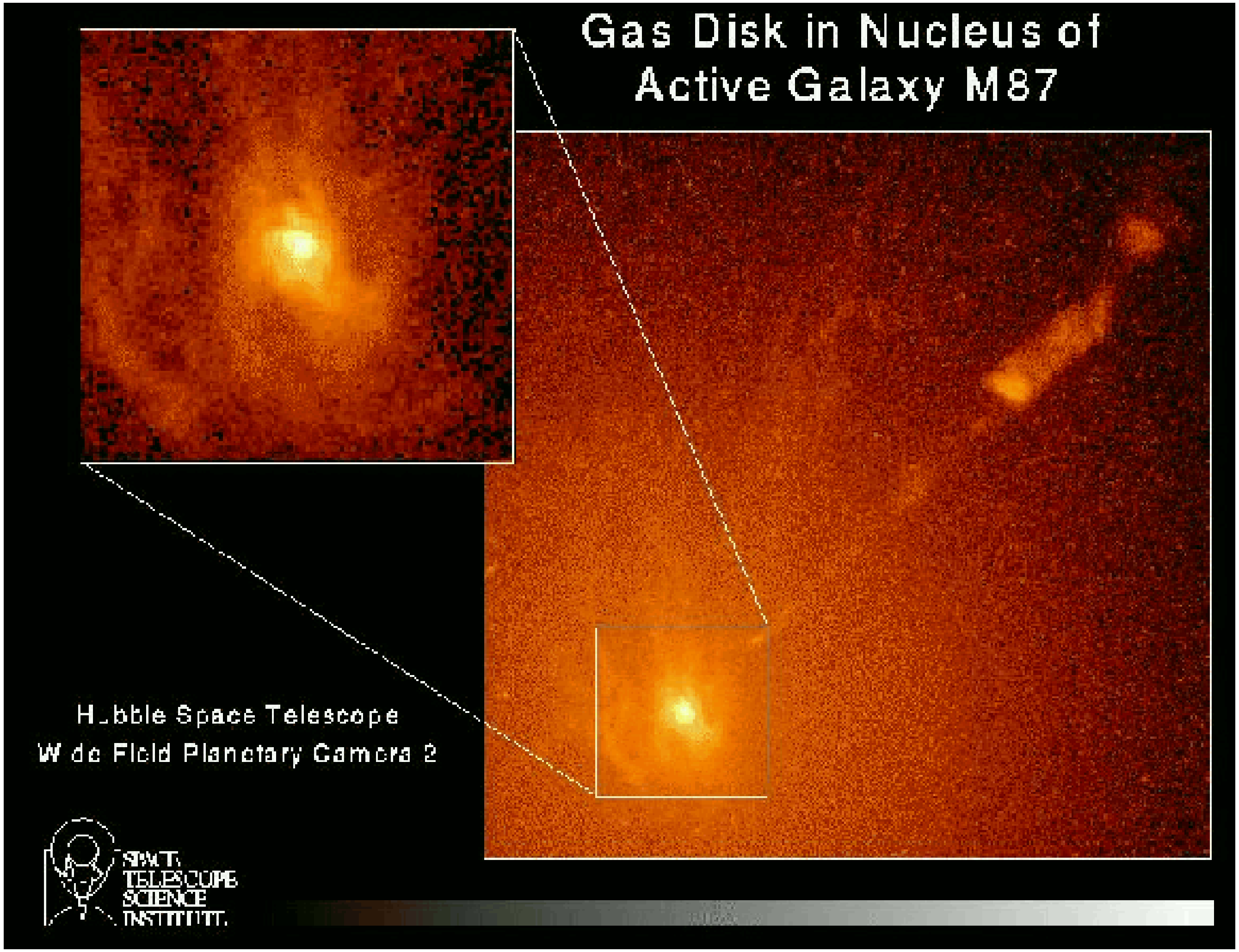
- By observing light from M87 with the Hubble telescope astronomers have determined that the orbital speed at a distance of 5.7×10^{17}m is 7.5×10^{5}m/s. What is the mass, m, of the object at the center of this galaxy?
- Here we just manipulate our equation to solve for the mass.

$$v = \sqrt{\frac{Gm}{r}} \rightarrow m = \frac{v^2 r}{G}, \text{substituting}$$

$$m = \frac{(7.5 \times 10^5\, m/s)^2 (5.7 \times 10^{17}\, m)}{6.67 \times 10^{-11}\, Nm^2/kg^2}$$

$$m = 4.8 \times 10^{39}\, kg \approx 2 \times 10^9 \text{solar masses!}$$

It turns out there are very few luminous stars in the center of M87, so something very heavy and dark lies at the center - a black hole?


Gas Disk in Nucleus of
Active Galaxy M87
Hubble Space Telescope
Wide Field Planetary Camera 2
SPACE
TELESCOPE
SCIENCE
INSTITUTE

Example: Characterizing Distant Satellites

- A satellite of mass m=5850kg is in circular orbit 4.1x10^{5}m above a planet of radius 4.15x10^{6}m. The period of the orbit is 2 hours. What is the true weight of the satellite when at rest on the surface of the planet?
- This is an interesting problem. We need to use our equation to find the mass of the planet and the use the gravitational law to measure the force or weight at the surface.

$$r = r_{Planet} + h = 4.15\times10^6\,m + 0.41\times10^6\,m = 4.56\times10^6\,m.$$

$$\text{Also } v = \frac{2\pi r}{T} = \frac{2\pi(4.56\times10^6\,m)}{7200s} = 3.98\times10^3\,m/s$$

$$v = \sqrt{\frac{Gm_E}{r}} \to m_{Planent} = \frac{v^2 r}{G} =$$

$$\frac{(3.98\times10^3\,m/s)^2(4.56\times10^6\,m)}{(6.67\times10^{-11}\,Nm^2/kg^2)} = 1.08\times10^{24}\,kg$$

So at the surface of the planet the force on the satellite or its weight would be

$$w = Gm_{Planet}m/r^2{}_{Planet} =$$

$$\frac{(6.67\times10^{-11}\,Nm^2/kg^2)(1.08\times10^{24}\,kg)(5.85\times10^3\,kg)}{(4.15\times10^6\,m)^2}$$

$$= 2.45\times10^4\,N \quad \text{Compare to } 5.73\times10^4\,N \text{ on earth.}$$

Synchronous Orbits

- Any satellite in an orbit locked over a specific location on earth (synchronous orbit), must revolve around the earth's center at the same angular velocity as the earth's surface.
- This only happens at a specific velocity and at a specific radius.
- Let's be a bit more quantitative.

We already have the relation $v = \sqrt{\frac{Gm_E}{r}}$,

The satellite's velocity can be related to the period with the simple relationship

$v = \frac{2\pi r}{T}$, equating these last two results:

$\frac{2\pi r}{T} = \sqrt{\frac{Gm_E}{r}}$ and solving or the radius:

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{Gm_E}{r} \rightarrow \boxed{r^3 = \frac{Gm_E T^2}{4\pi^2}}$$

The Radius of a Synchronous Orbit

- What is the height above the earth's surface at which all synchronous satellites must be placed in orbit?
- Our last result gives the relations ship between the radius and the period of an orbit.
- If we simply set the period equal to one day we have radius of a synchronous orbit.

$$r^3 = \frac{Gm_E T^2}{4\pi^2} \text{ but for one day T} = 86{,}400\text{s}$$

$$r^3 = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(5.98\times10^{24}\,kg)(8.64\times10^4)^2}{4\pi^2}$$

$$r = 4.23\times10^7\,m = 42{,}300km.$$

But remember this is the distance form the center of the earth to the satellite, the height is given by,

$$\text{H} = 42{,}300\text{km} - 6{,}400\text{km} = 35{,}900\text{km} = 22{,}300 \text{ miles}$$

Schedule

- Next Quiz: Wednesday March 7^{th}
- Units 3 and 4:
  - Force and Laws of Motion
  - Circular Motion and Gravity.
- Homework and Extra Credit for Units 3 and 4 due before the test.

Quiz Details

- Same format as first quiz
- Calculators OK, but not necessary
- Total of 50 points
  - 4 multiple choice @ 5 pts each
  - 3 problems @ 5, 10, 15 points.
- Friday you'll hear what is on the test!