Status: Unit 4 - Circular Motion and Gravity

- ✓ Uniform circular motion, angular variables, and the equations of motion for angular motion (3-9, 10-1, 10-2)
- Applications of Newton's Laws to circular motion (5-2, 5-3, 5-4)
- Harmonic Motion (14-1, 14-2)
- The Universal Law of Gravitation, Satellites, and Kepler's Law's (Chapter 6)

Overview of Important Results

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

$$\overline{\omega} = \frac{\omega_0 + \omega}{2}$$

$$v = \omega R$$

$$a_R = \frac{v^2}{R} = \omega^2 R$$

$$a_{\tan} = R\alpha$$

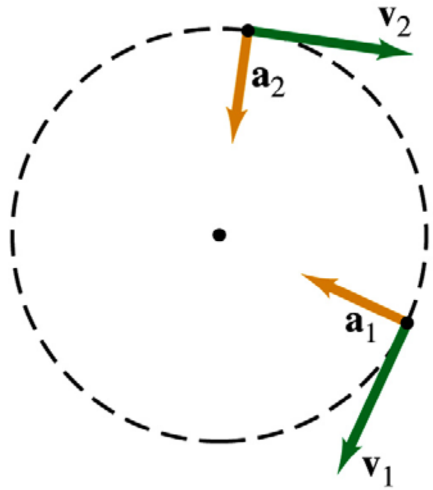
$$\omega = 2\pi f$$

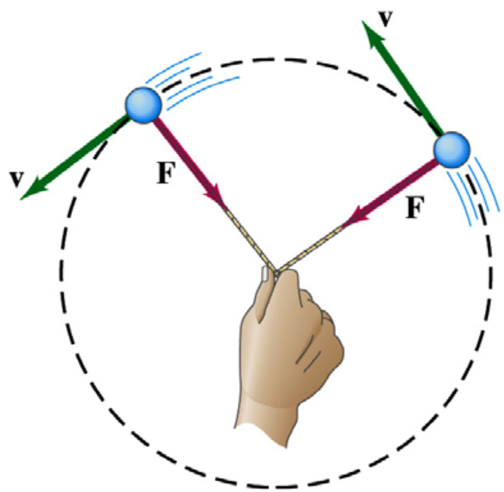
$$T = \frac{1}{f}$$

Dynamics of Uniform Circular Motion

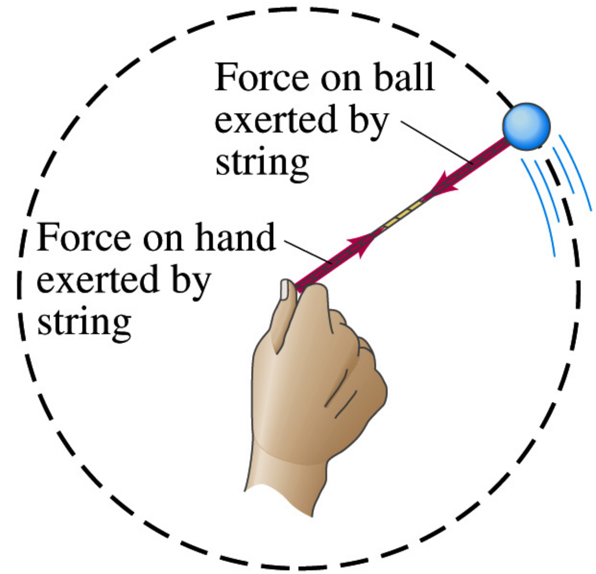
- Consider the centripetal acceleration $\mathbf{a}_R$ of a rotating mass:
  - The magnitude is constant.
  - The direction is perpendicular to the velocity and inward.
  - The direction is continually changing.
- Since $\mathbf{a}_R$ is nonzero, according to Newton's 2nd Law, there must be a force involved.

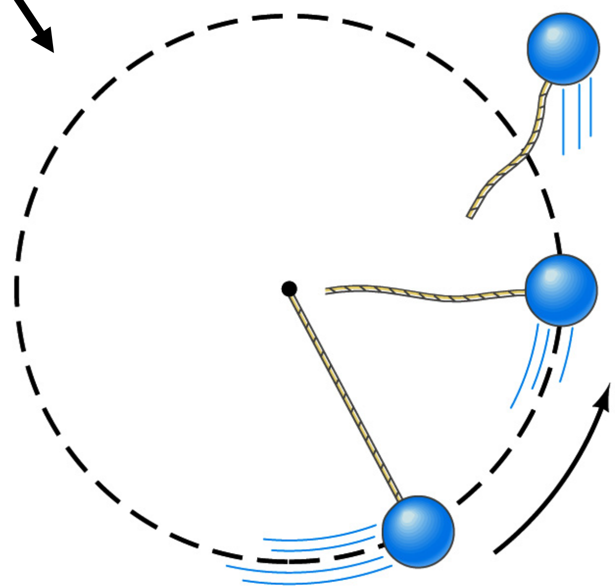
$$\boxed{\Sigma F_R = m a_R = m\frac{v^2}{R}}$$



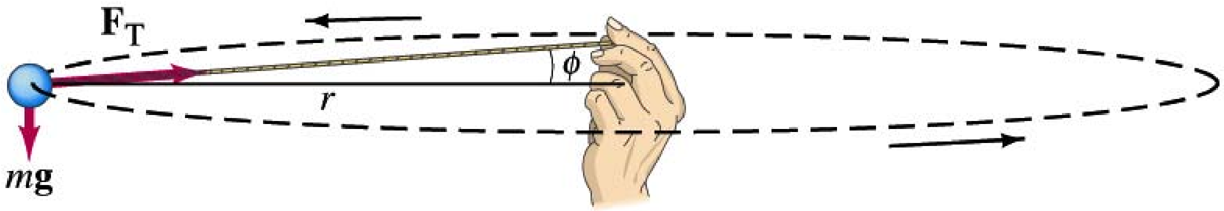


- Consider a ball on a string:
  - There must be a net force force in the radial direction for it to move in a circle.
  - Other wise it would just fly out along a straight line, with unchanged velocity as stated by Newton's 1st Law
- Don't confuse the outward force on your hand (exerted by the ball via the string) with the inward force on the ball (exerted by your hand via the string).
- That confusion leads to the mis-statement that there is a "centrifugal" (or center-fleeing) force on the ball. That's not the case at all!

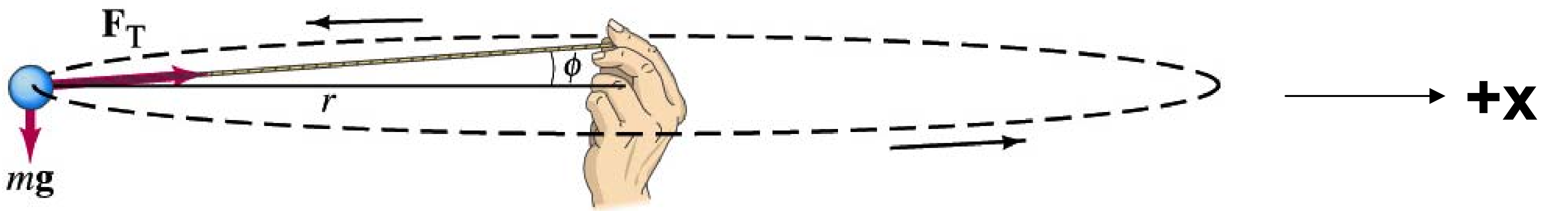




Example 1: Force on a Revolving Ball



- As shown in the figure, a ball of mass 0.150 kg fixed to a string is rotating with a period of T=0.500s and at a radius of 0.600 m.
- What is the force the person holding the ball must exert on the string?



- As usual we start with the free-body diagram.
- Note there are two forces
  - gravity or the weight, m**g**
  - tensional force exerted by the string, $\mathbf{F_T}$
- We'll make the approximation that the ball's mass is small enough that the rotation remains horizontal, ϕ=0. (This is that judgment aspect that's often required in physics.)

- Looking at just the x component then we have a pretty simple result:

$$\Sigma F_X = ma_X \rightarrow$$

$$\Sigma F_X = m\frac{v^2}{r} \rightarrow$$

$$\Sigma F_X = m\frac{(2\pi r/T)^2}{r} \rightarrow$$

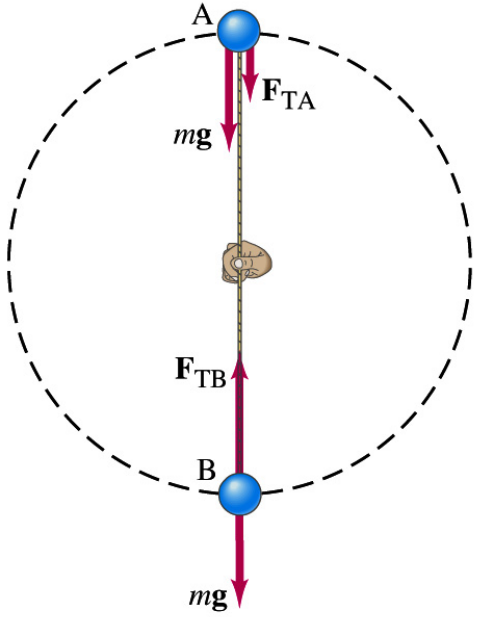
$$F = \frac{4\pi^2 mr}{T^2} =$$

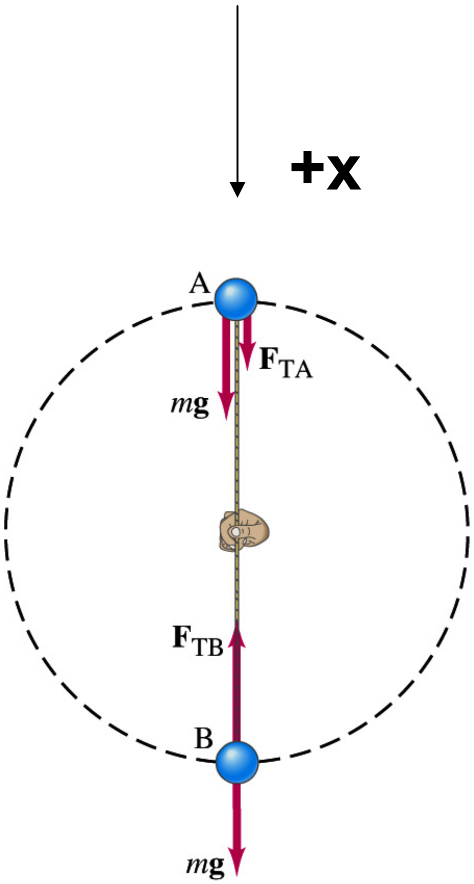
$$\frac{4\pi^2(0.15kg)(0.60m)}{(0.50s)^2}$$

$$\approx 14N$$

Example 2: A Vertically Revolving Ball

- Now lets switch the orientation of the ball to the vertical and lengthen the string to 1.10 m.
- For circular motion (constant speed and radius), what's the speed of the ball at the top?
- What's the tension at the bottom if the ball is moving twice that speed?





- So to the free-body diagram, at the top, at point A, there are two forces:
  - tensional force exerted by the string, $\mathbf{F}_{TA}$
  - gravity or the weight, $m\mathbf{g}$
- In the x direction:

$$\Sigma F_R = ma_R = m\frac{v_A^2}{r}$$

$$\Sigma F_R = F_{TA} + mg$$

$$m\frac{v_A^2}{r} = F_{TA} + mg$$

- Let's talk about the dependencies of this equation. Since mg is constant, the tension will be larger should v_A increase. This seems intuitive.
- Now the ball will fall if the tension vanishes or if F_{TA} is zero

$$m\frac{v_A^2}{r} = 0 + mg \rightarrow$$

$$v_A = \sqrt{gr} =$$

$$\sqrt{9.80m/s^2 \times 1.10m} =$$

$$3.28m/s$$

- At point B there are also two forces but both acting in opposite directions. Using the same coordinate system.

$$\Sigma F_R = ma_R = m\frac{v_B^{\,2}}{r} \rightarrow$$

$$\Sigma F_R = F_{TB} - mg \rightarrow$$

$$m\frac{v_B^{\,2}}{r} = F_{TB} - mg \rightarrow$$

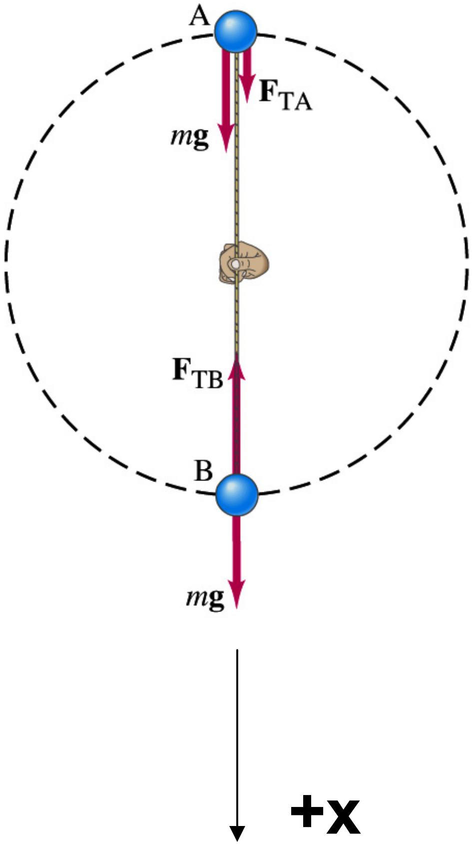
$$F_{TB} = m(\frac{v_B^{\,2}}{r} + g)$$

Now since we were given $v_B = 6.56 m/s$,

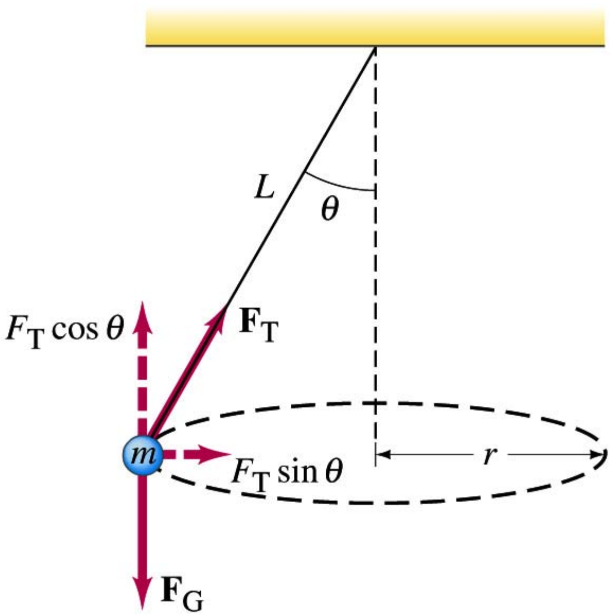
$$F_{TB} = 0.150kg(\frac{(6.56m/s)^2}{1.10m} + 9.80m/s^2)$$

$$F_{TB} = 7.34N$$

- Note that the tension still provides the radial acceleration but now must also be larger than ma_R to compensate for gravity.



Example 3: The Conical Pendulum



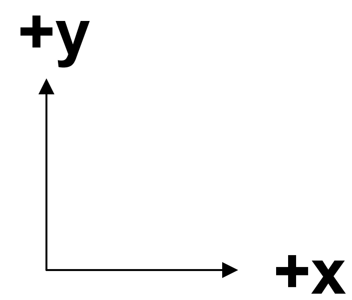
- Here we have a small mass m hanging from a cord of length Lat an angle θ with the vertical.
- The ball is revolving in a circle as shown a radius r=Lsinθ
- What is the origin and direction of the mass acceleration
- Calculate the speed v and period of revolution T for the object in terms of L, θ, g, and m.
- We assume that
  – this pendulum is in uniform circular motion
  – the vertical position does not change.
  – there is no friction

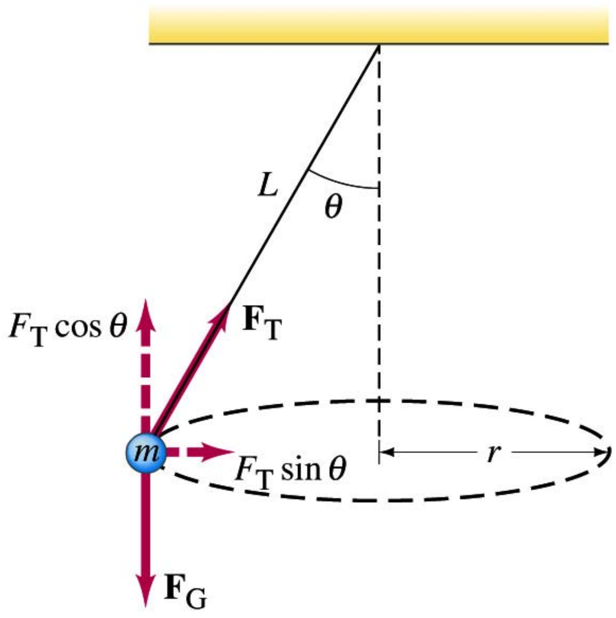
- Of course we turn to the free body diagram and apply Newton's 2nd law.
- There are two forces, and only two, since there is no friction
  - The weight m**g**
  - The tension, $\mathbf{F_T}$

  That's it!
- In the vertical direction the 2nd Law gives:

  $$F_T\cos\theta - mg = 0$$
- In the horizontal direction there is one force $F_T\sin\theta$ but since we have uniform circular motion the 2nd law tells us:

  $$F_T\sin\theta = mv^2/r$$





We were asked to find v and T. At this point we have three equations, that will be enough:

$$F_T \cos\theta = mg\ ,\ \ F_T \sin\theta = m\frac{v^2}{r}, \text{ and } r = L\sin\theta$$

Solving the second for v we find:

$$v = \sqrt{\frac{rF_T \sin\theta}{m}}$$ where we take the positive, physical root.

We can substitute for F_T using the first equation and for r using the third,

$$v = \sqrt{\frac{L\sin\theta(\frac{mg}{\cos\theta})\sin\theta}{m}} = \sqrt{\frac{Lg\sin^2\theta}{\cos\theta}}$$

Note there is no dependence on the mass, only L, g, θ.

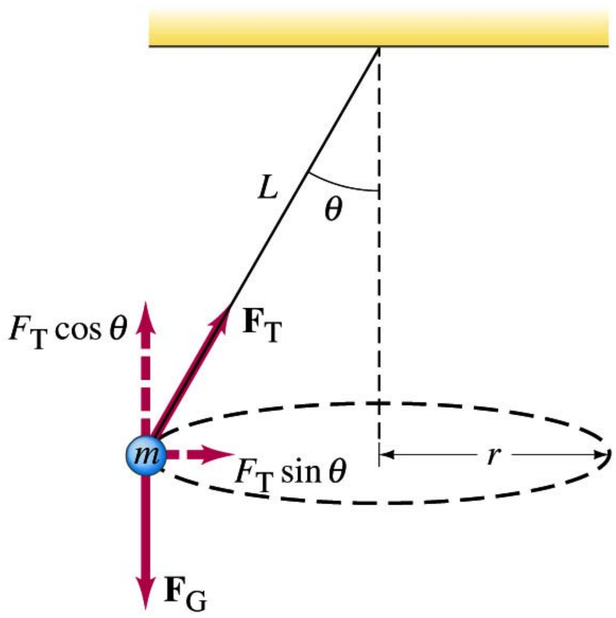
What about the period? Well we can relate it the the derived velocity by noting that $v = 2\pi r / T$.

We just derived the velocity, lets turn this around and substitute r and v into $T = \dfrac{2\pi r}{v}$

$$T = \frac{2\pi L \sin\theta}{\sqrt{\dfrac{Lg\sin^2\theta}{\cos\theta}}} = 2\pi\sqrt{\frac{L^2\sin^2\theta}{\dfrac{Lg\sin^2\theta}{\cos\theta}}} =$$

$$T = 2\pi\sqrt{\frac{L}{\dfrac{g}{\cos\theta}}} = 2\pi\sqrt{\frac{L}{g}\cos\theta}$$

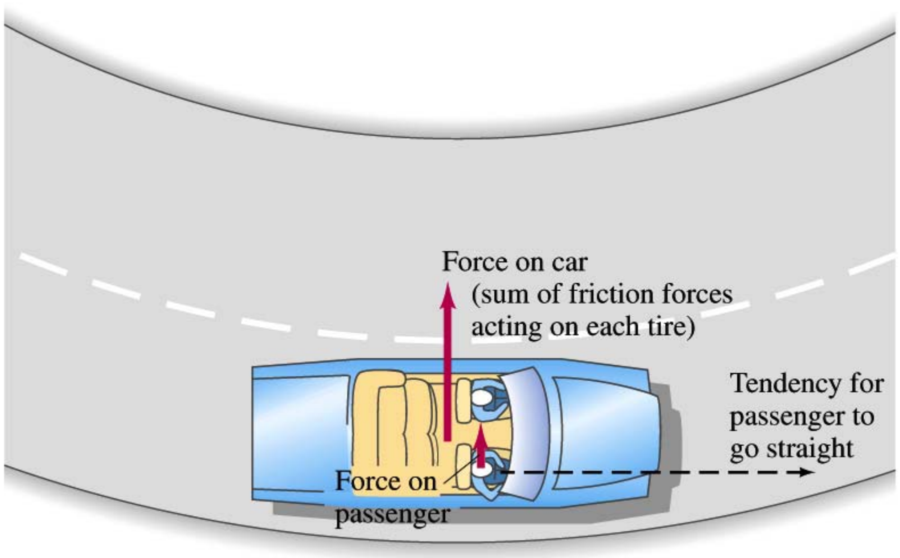
Again the period does not depend on the mass, only L, g, and θ.



$$v = \left(\sqrt{Lg}\right)\sqrt{\frac{\sin^2\theta}{\cos\theta}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}\sqrt{\cos\theta}$$

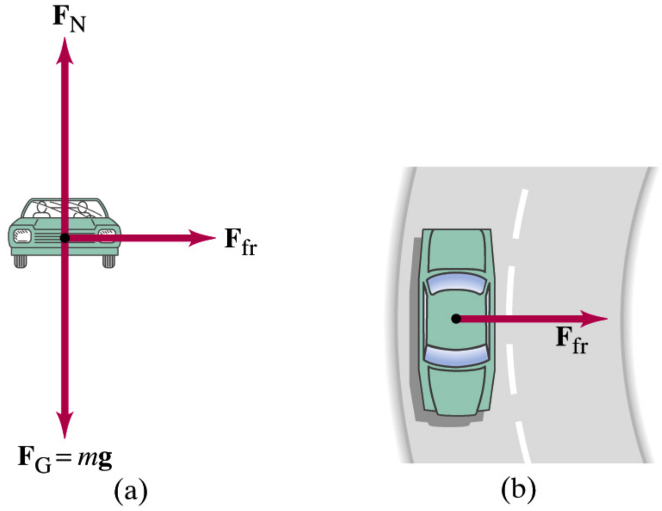
Designing Your Highways!

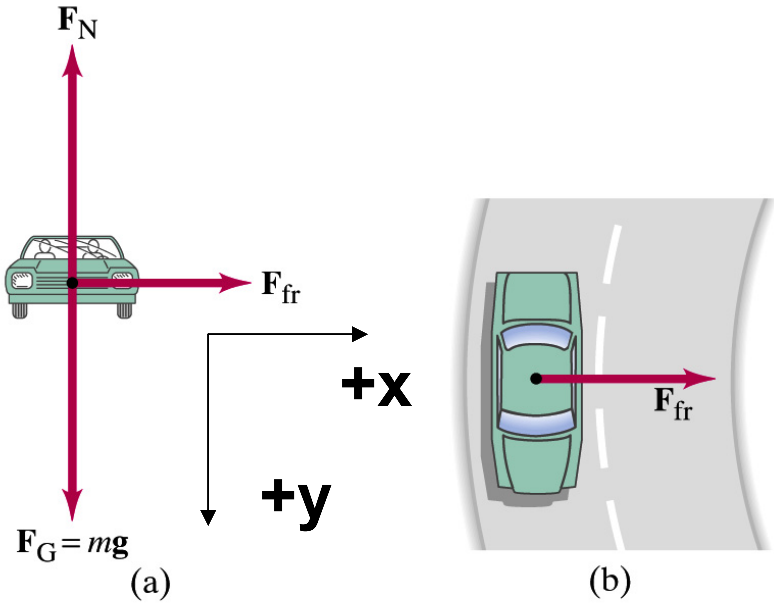


- Turns out this stuff is actually useful for
  - civil engineering such as road design
  - A NASCAR track
- Let's consider a car taking a curve, by now it's pretty clear there must be a centripetal forces present to keep the car on the curve or, more precisely, in uniform circular motion.
- This force actually comes from the friction between the wheels of the car and the road.
- Don't be misled by the outward force against the door you feel as a passenger, that's the door pushing you inward to keep YOU on track!

Example 4: Analysis of a Skid

- The setup: a 1000kg car negotiates a curve of radius 50m at 14 m/s.
- The problem:
  - If the pavement is dry and μ_s=0.60, will the car make the turn?
  - How about, if the pavement is icy and μ_s=0.25?





- Looking at the car head-on the free-body diagram shows three forces, gravity, the normal force, and friction.
- We see only one force offers the inward acceleration needed to maintain circular motion - friction.

- First off, in order to maintain uniform circular motion the centripetal force must be:

$$\Sigma F_R = ma_R = m\frac{v^2}{r} =$$

$$1000kg \times \frac{(14m/s)^2}{50m} =$$

$$3900N$$

- To find the frictional force we start with the normal force, from Newton's second law:

$$\Sigma F_y = 0 = F_N - mg \rightarrow$$

$$F_N = mg = 1000kg \times 9.8m/s^2$$

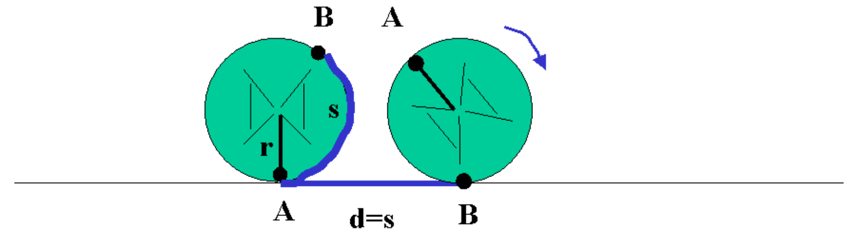
$$= 9800N$$

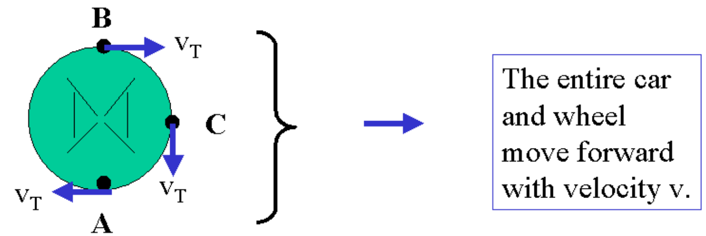
- At the point a wheel contacts the road the relative velocity between the wheel and road is zero.
- Proof:
  - From the top figure we see that when a rolling wheel travels through arc "s" the wheel and car move forward distance "d", so s=d.
  - If we divide both by t, the time of the roll and the translation forward, we get

$$s/t = d/t \rightarrow$$

$$v_T = v_{CAR}$$

  - Thus a point on the wheel is moving forward with the same velocity as the car, v_{CAR}, while rotating about its axis





  - For point B the total velocity is just the addition

$$v_{Total} = v_T + v_{CAR} \rightarrow$$

$$v_{Total} = v_{CAR} + v_{CAR} = 2v_{CAR}$$

  - And for point A

$$v_{Total} = -v_T + v_{CAR} \rightarrow$$

$$v_{Total} = -v_{CAR} + v_{CAR} = 0!$$

- Back to the analysis of a skid.
- Since v=0 at contact, if a car is holding the road, we can use the static coefficient of friction.
- If it's sliding, we use the kinetic coefficient of friction.
- Remember, we need 3900N to stay in uniform circular motion.

- Static friction force first:

$$F_{fr}(\max) = \mu_s F_N =$$

$$0.60 \times 9800N = 5900N$$

Holds the road!

- Now kinetic,

$$F_{fr} = \mu_K F_N =$$

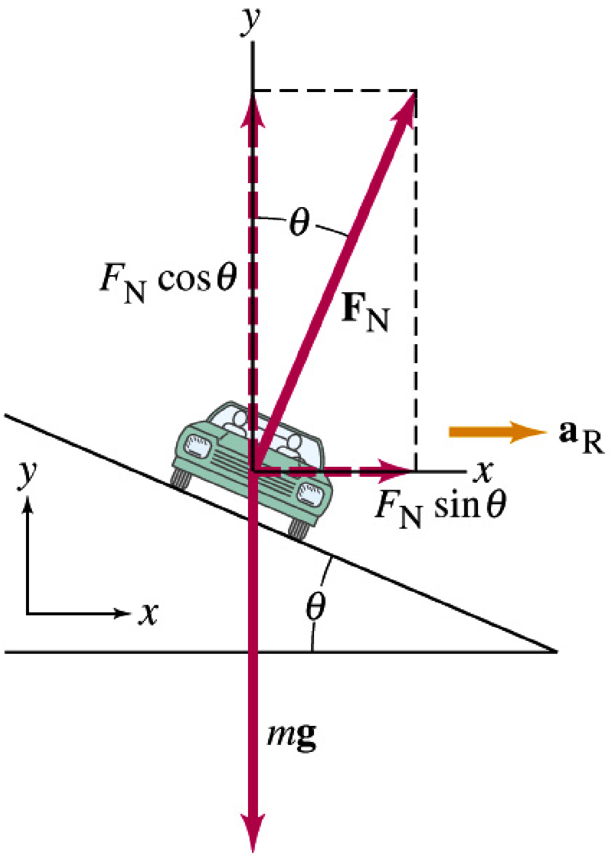
$$0.25 \times 9800N = 2500N$$

Off it goes!

The Theory of Banked Curves

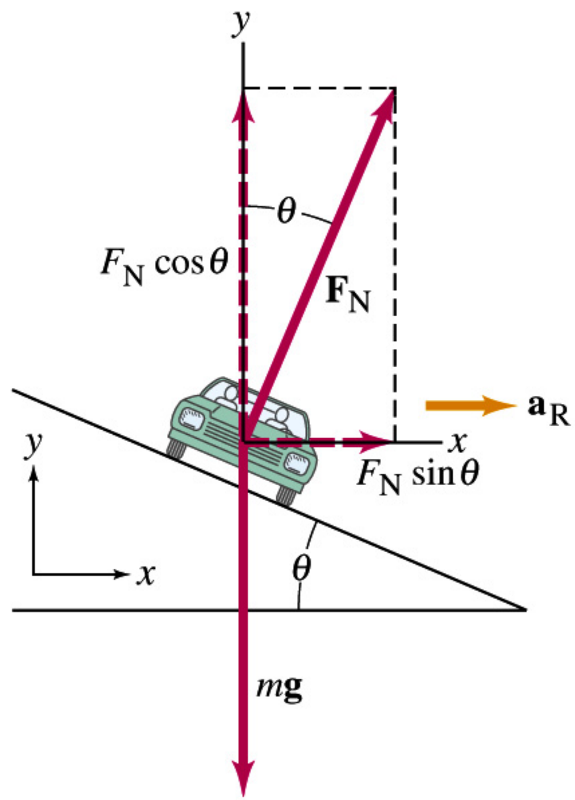
- The Indy picture shows that the race cars (and street cars for that matter) require some help negotiating curves.
- By banking a curve, the car's own weight, through a component of the normal force, can be used to provide the centripetal force needed to stay on the road.
- In fact for a given angle there is a maxinum speed for which no friction is required at all.
- From the figure this is given by

$$F_N \sin\theta = m\frac{v^2}{r}$$



Example 5: Banking Angle

- Problem: For a car traveling at speed v around a curve of radius r, what is the banking angle θ for which no friction is required? What is the angle for a 50km/hr (14m/s) off ramp with radius 50m?
- To the free-body diagram! Note that we've picked an unusual coordinate system. Not down the inclined plane, but aligned with the radial direction. That's because we want to determine the component of any force or forces that may act as a centripetal force.
- We are ignoring friction so the only two forces to consider are the weight m**g** and the normal force $\mathbf{F_N}$. As can be seen only the normal force has an inward component.



- As we discussed earlier in the horizontal or + x direction, Newton's 2nd law leads to:

$$F_N \sin\theta = m\frac{v^2}{r}$$

- In the vertical direction we have:

$$\Sigma F_y = F_N \cos\theta - mg \rightarrow$$

$$F_N \cos\theta - mg = 0$$

Since the acceleration in this direction is zero, solving for F_N

$$F_N = \frac{mg}{\cos\theta}$$

- Note that the normal force is greater than the weight.

- This last result can be substituted into the first:

$$\frac{mg}{\cos\theta}\sin\theta = m\frac{v^2}{r} \rightarrow$$

$$mg\tan\theta = m\frac{v^2}{r} \rightarrow$$

$$g\tan\theta = \frac{v^2}{r} \rightarrow$$

$$\tan\theta = \frac{v^2}{gr}$$

- For v=14m/s and r= 50m

$$\tan\theta = \frac{v^2}{gr} = \frac{(14m/s)^2}{9.8m/s^2 \times 50m} = 0.40$$

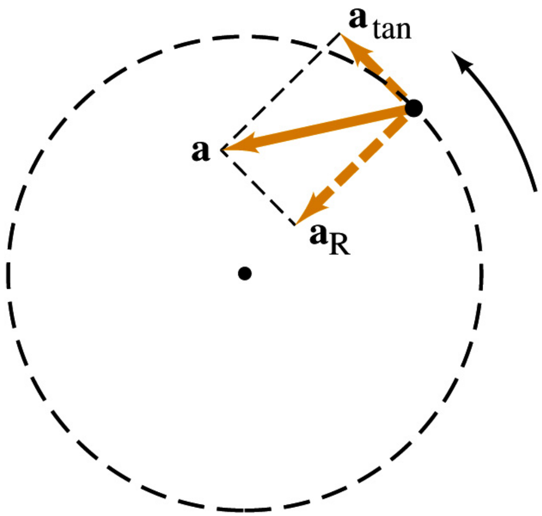
$$\theta = 22^o$$

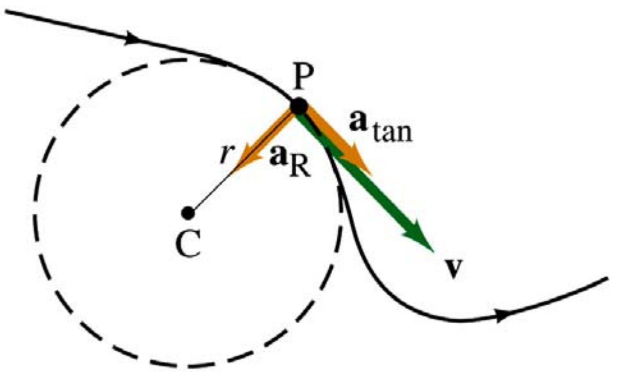
Non-uniform circular motion

- More specifically lets consider constant r, but changing speed.
- This means there will be a tangential acceleration with magnitude given by $a_{\tan} = \frac{dv}{dt}$
- But the radial acceleration remains: $a_R = m\frac{v^2}{r}$
- These two vectors are always perpendicular, so the total acceleration has magnitude:

$$a = \sqrt{a_{\tan}{}^2 + a_R^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(m\frac{v^2}{r}\right)^2}$$

- This notion can actually be generalized to any circular portion of a trajectory





Example 6: The Anne-Glidden Exit

- When taking the Anne Glidden exit your speed drops from 65 mph (30m/s) to 30 mph (13m/s) in 5.0 seconds. The radius of the curve is 500 m.
- What is your average tangential deceleration and your radial acceleration at the beginning and end of your exit?
- Well the average deceleration is just given in the usual way

$$\overline{a}_{\tan} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{13m/s - 30m/s}{5.0s} =$$
$$-3.4m/s^2$$

- The radial acceleration is

$$a_R = \frac{v^2}{r}$$

At the start of the curve

$$a_R = \frac{(30m/s)^2}{500m} = 1.8m/s^2$$

At the end of the curve

$$a_R = \frac{(13m/s)^2}{500m} = 0.3m/s^2$$

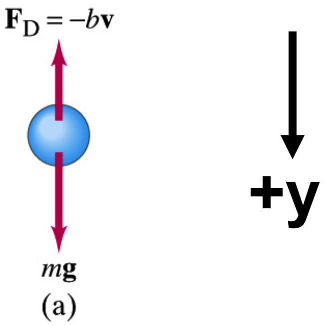
A small diversion

- The equation relating the total acceleration to a function of the derivative of velocity and the velocity squared is quite common in form.
- In fact velocity-dependent forces are not unusual. Terminal velocity which we've already discussed is a good example.
- An object falling in a liquid is quite interesting. Here the object is subject to a drag force from friction which is proportion and opposite the velocity of the object or F_D=-bv
- Here Newton's 2nd Law gives

$$\Sigma F_y = mg - bv \rightarrow$$

$$mg - bv = ma \rightarrow$$

$$mg - bv = m\frac{dv}{dt}$$



$$mg - bv = m\frac{dv}{dt} \rightarrow$$

$$\frac{dv}{dt} = g - \frac{b}{m}v \rightarrow$$

$$\frac{dv}{g - \frac{b}{m}v} = dt \rightarrow$$

$$\frac{dv}{v - \frac{mg}{b}} = -\frac{b}{m}dt$$

But this can be integrated from $\mathrm{v} = 0$ at $\mathrm{t} = 0$:

$$\int_{o}^{v} \frac{dv}{g - \frac{b}{m}v} = \int_{0}^{t} -\frac{b}{m}dt \rightarrow$$

$$\int_o^v \frac{dv}{g - \frac{b}{m}v} = \int_0^t -\frac{b}{m}dt \rightarrow$$

$$\ln\left(v - \frac{mg}{b}\right) - \ln\left(-\frac{mg}{b}\right) = -\frac{b}{m}t \rightarrow$$

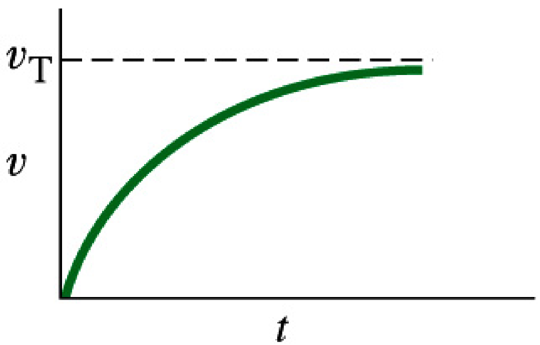
$$\ln\left(\frac{v - \frac{mg}{b}}{-\frac{mg}{b}}\right) = -\frac{b}{m}t$$

Raising both side to the exponential

$$\left(\frac{v - \frac{mg}{b}}{-\frac{mg}{b}}\right) = e^{-\frac{b}{m}t} \rightarrow$$

$$v = \frac{mg}{b}(1 - e^{-\frac{b}{m}t})$$

- Note the limits
  - At t=0, v=0
  - At t=infinity, v=mg/b



(b)

Summary

- For uniform circular motion the key notion involves a combination of Newton's 2nd Law with the geometric observation that $a_R = v^2/r$.
- We've explored the physics of a ball on a string, a conical pendulum, and banked curves.
- We've also taken a look at how to handle circular motion at constant radius but changing speed.
- Next we'll do something a bit different and discuss harmonic motion, although not strictly "circular" it's got much in common…and for your future reference it turns out to be one of the most instructive points of contact between classical and quantum physics.