Add Some Realism to our Ideal Accelerator

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- Steering (dipole) Errors
- Focusing (quadrupole) Errors
- Errors creating Linear Coupling
- Chromatic (momentum) Effects
- Nonlinear Motion and Resonances

- Not only will errors create perturbations in the beam size, etc., but they will also tend to identify operational considerations, such as frequency choices, corrector placement, alignment tolerances, power supply specifications, etc.

Steering (dipole) Errors

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- dipole field error:
  - manufacturing; powering; control setting, …

$$B_y = B_0 \qquad \longrightarrow \qquad B_y = B_0 + \Delta B$$

$$\Delta x' = -\frac{\Delta B \ell}{B\rho}$$

- dipole field “roll” (about the longitudinal axis)

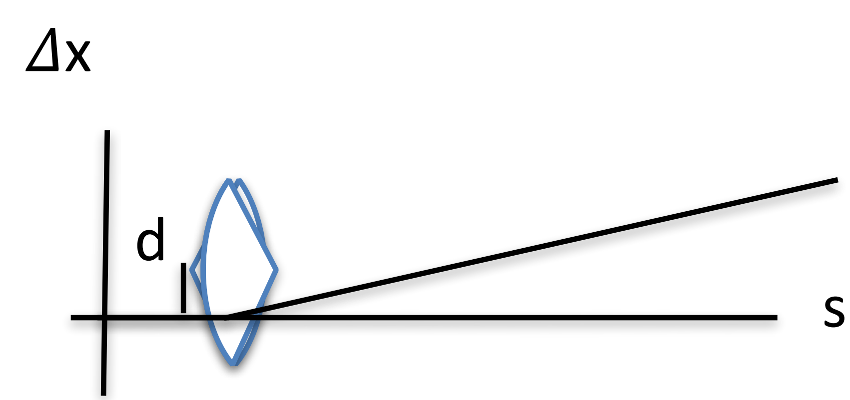
$$B_y = B_0, \quad B_x = 0$$

$$\begin{aligned} B_y &= B_0 \cos\phi \approx B_0 \\ B_x &= B_0 \sin\phi \approx \phi B_0 \end{aligned}$$

$$\Delta y' = \phi \, \frac{B_0 \ell}{B\rho} = \phi \, \theta_0$$

- Quadrupole misalignment:

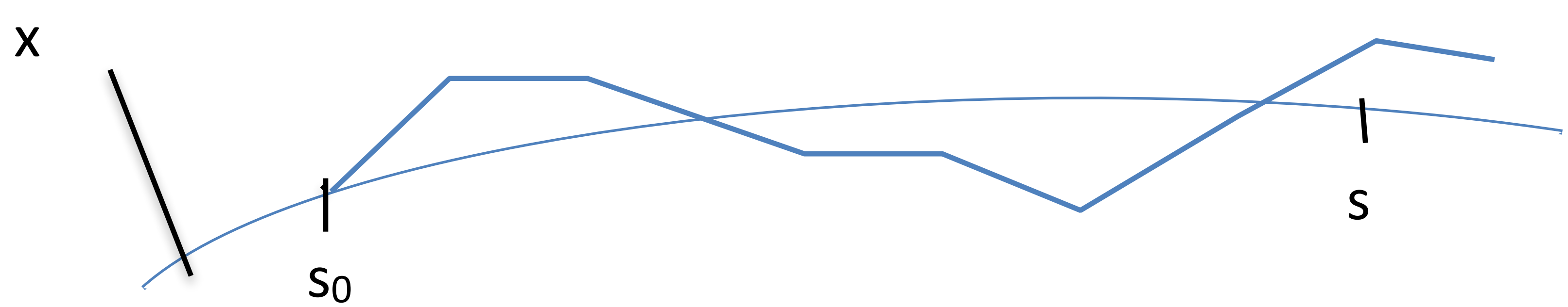
$$\Delta x' = \frac{d}{F}$$



Steering (dipole) Errors

- A field error creates a betatron oscillation…

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



due to the small error field:

$$\Delta x' = x'_0 = \Delta\theta$$

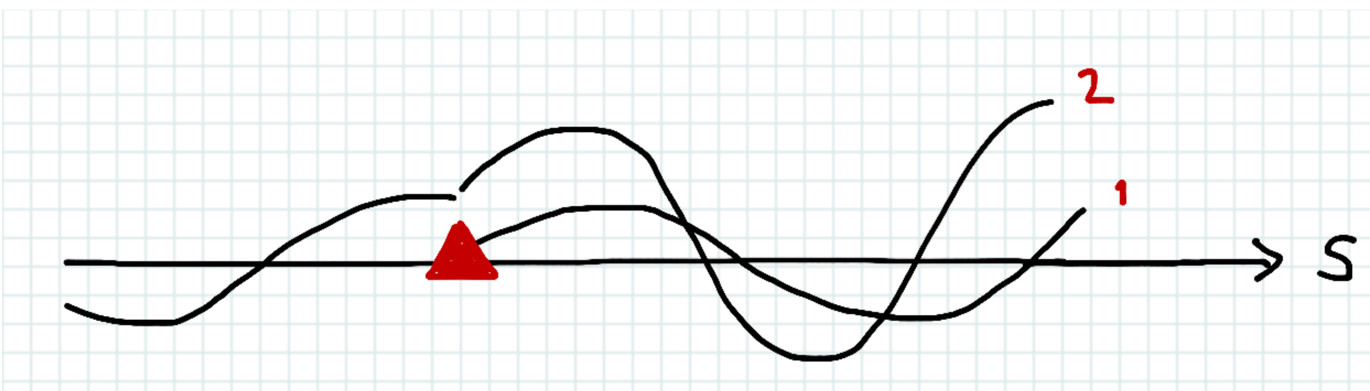
$$\Delta x = x_0 = 0$$

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi$$

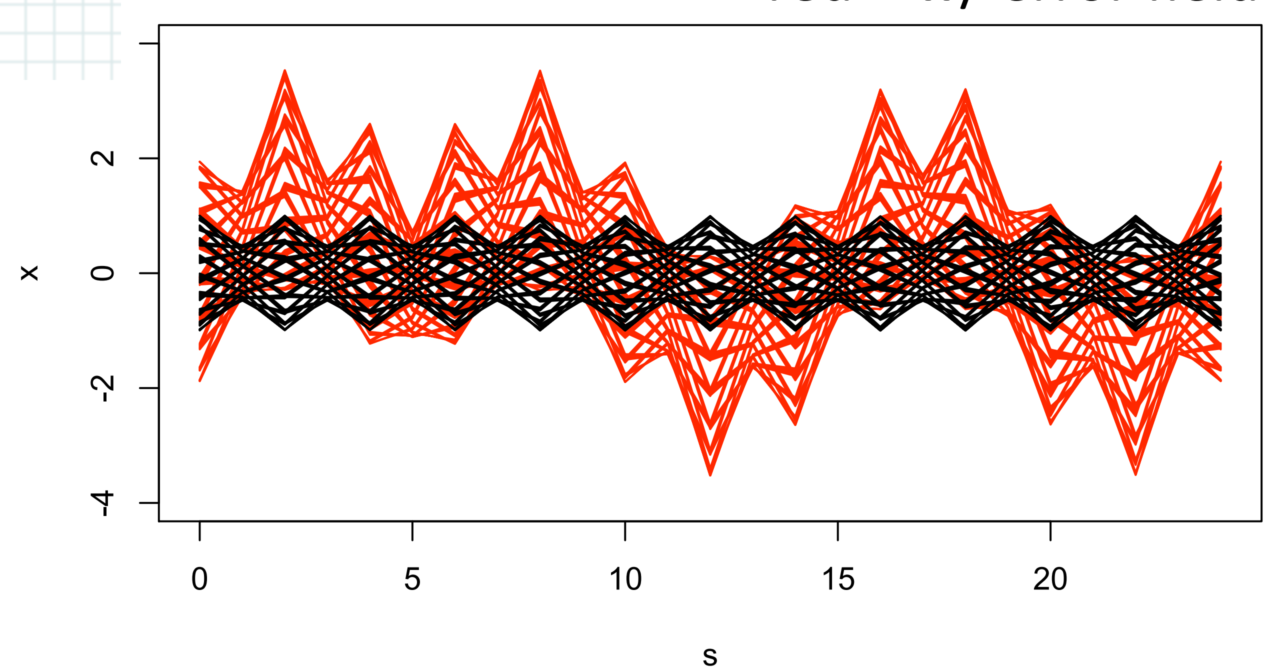
Steering (dipole) Errors

- Closed orbit distortions in a circular accelerator
  - These are not "one-time" kicks; they affect the particle motion every revolution

see ClosedOrbit.R



black = nominal
red = w/ error field



The trajectory of each particle will be altered by the angle $\Delta\theta$ every time it passes through the error field

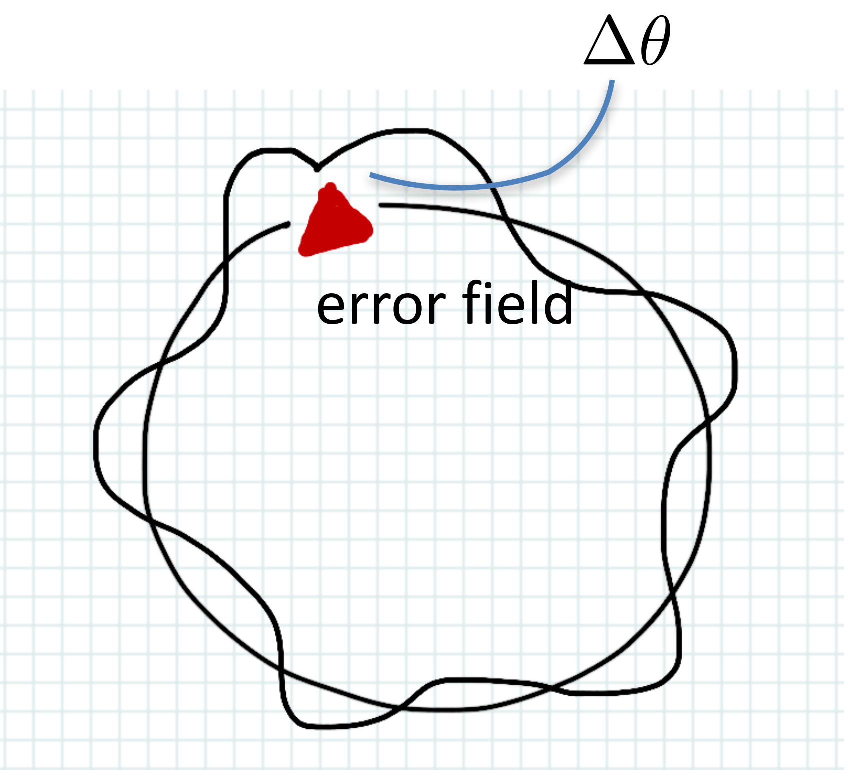
The Closed Orbit

- Want to find the one trajectory which, upon passing through the error field, will come back upon itself
  - this is the "closed" trajectory, or closed orbit

$$M_0 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

- When find x0, x'0, can find x,x' downstream:

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M_0)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$



$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}} \sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}} \cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2} (\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Closed Orbit Distortion from Single Error

$$
\begin{aligned}
\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} &= (I - M_0)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \\
&= (I - e^{J_0\mu})^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \\
&= \left[e^{J_0\mu/2}(e^{-J_0\mu/2} - e^{J_0\mu/2}) \right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \\
&= \left[e^{J_0\mu/2}(-2J_0 \sin\mu/2) \right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \\
&= -\frac{1}{2\sin\pi\nu} J_0^{-1} e^{-J_0\mu/2} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \\
&= \frac{1}{2\sin\pi\nu} J_0 (I\cos\pi\nu - J_0 \sin\pi\nu) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \\
&= \frac{1}{2\sin\pi\nu} (I\sin\pi\nu + J_0\cos\pi\nu) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \\
&= \frac{\Delta\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}
\end{aligned}
$$

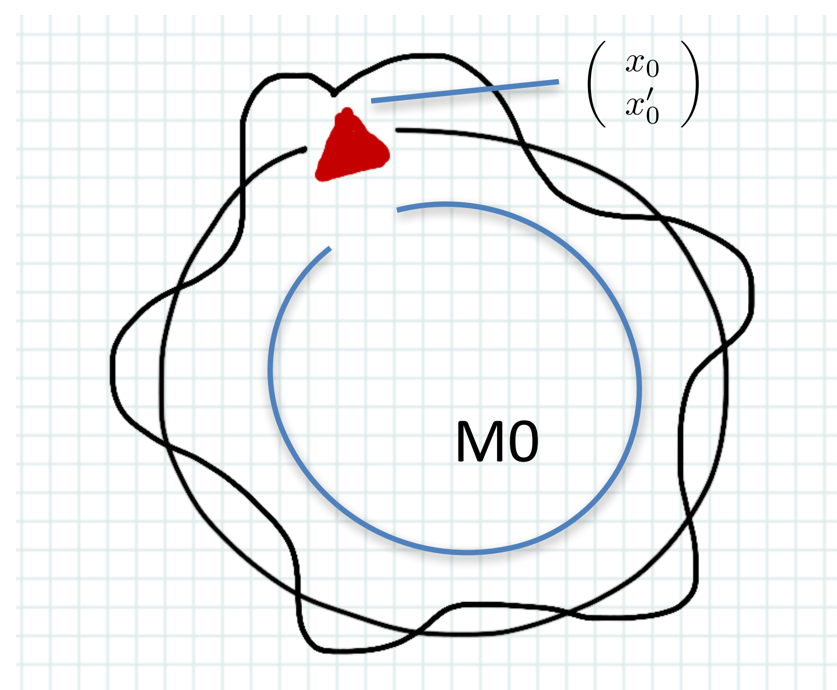
$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$M_0 = I\cos\mu + J_0 \sin\mu$$

$$M_0 = e^{J_0\mu} = e^{J_0 2\pi\nu}$$

$$J_0^2 = -I$$

$$\mu = 2\pi\nu$$



Closed Orbit Distortion from Single Error

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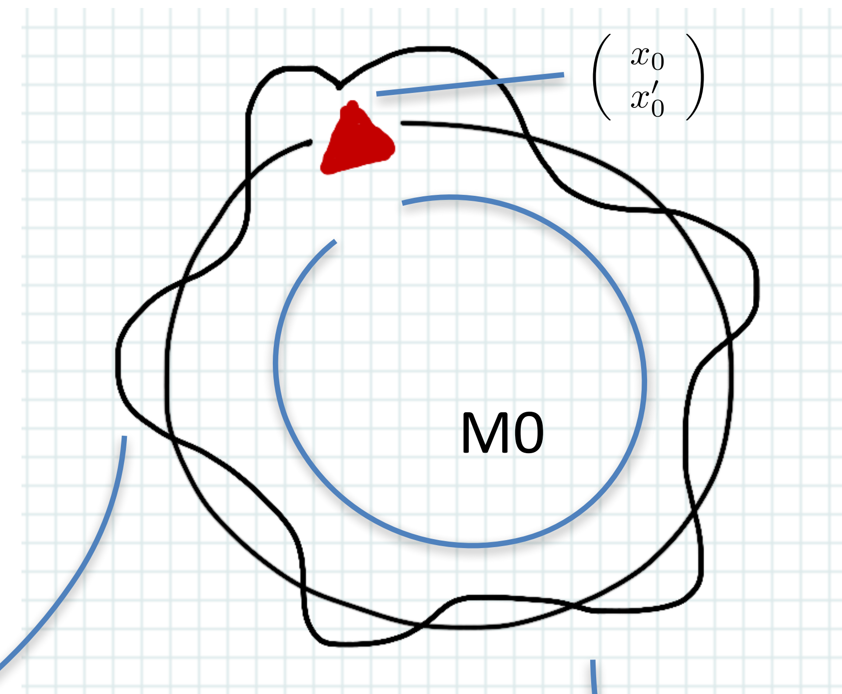
$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{\Delta\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$



$$\boxed{\Delta x(s) = \frac{\Delta\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}\cos\left[|\psi(s)-\psi_0| - \pi\nu\right]}$$

ν oscillations

as ν —> integer, huge distortions
a resonance!

If have a collection of errors about the accelerator, then at any one point:

$$\Delta x(s) = \sum_i \frac{\Delta\theta_i\sqrt{\beta_i\beta(s)}}{2\sin\pi\nu}\cos\left[|\psi(s)-\psi_i| - \pi\nu\right]$$

Trajectory/Orbit Correction

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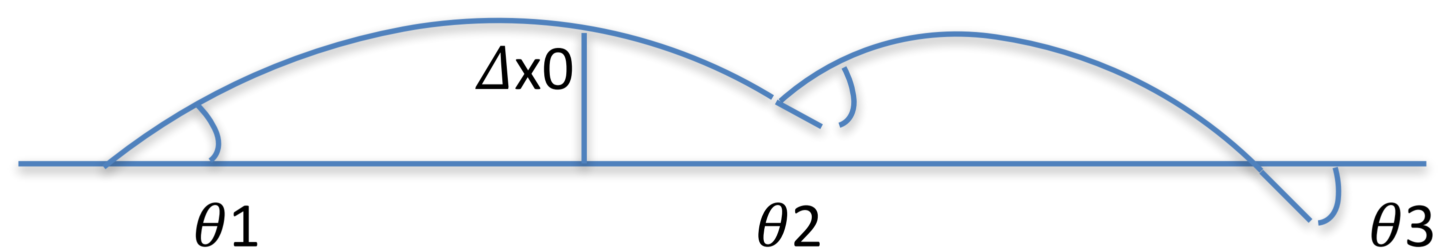
- To make a local adjustment or correction of the position of the beam in a beam line or synchrotron, three correctors are required (in general):



$$\theta_1 = \frac{\Delta x_0}{\sqrt{\beta_0\beta_1}\sin\psi_{10}}$$

$$\theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\,\frac{\sin\psi_{13}}{\sin\psi_{23}}$$

The trajectory before $\theta 1$ and after $\theta 3$ is left undisturbed

$$\theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

Orbit Corrections

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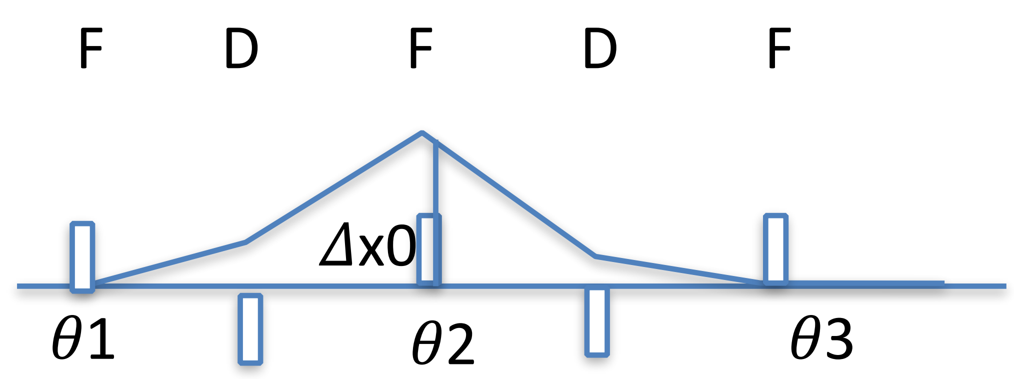
- As an example, in a "FODO" synchrotron, one would place correctors near the location of each quadrupole — at maximum beta locations, and at the source of likely steering errors (misaligned quads)

$$\psi_{13} = 2\psi_{12} = 2\psi_{23} = 2\mu$$



$$\theta_1 = \frac{\Delta x}{\hat{\beta}\sin\mu}$$

$$\theta_2 = -\theta_1 \frac{\sin 2\mu}{\sin\mu} \quad \Rightarrow \quad \theta_2 = -2\theta_1\cos\mu$$

$$\theta_3 = \theta_1$$

Alignment Specifications Discussion

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- see TrajTrace.Rmd
-

Focusing (gradient) Errors

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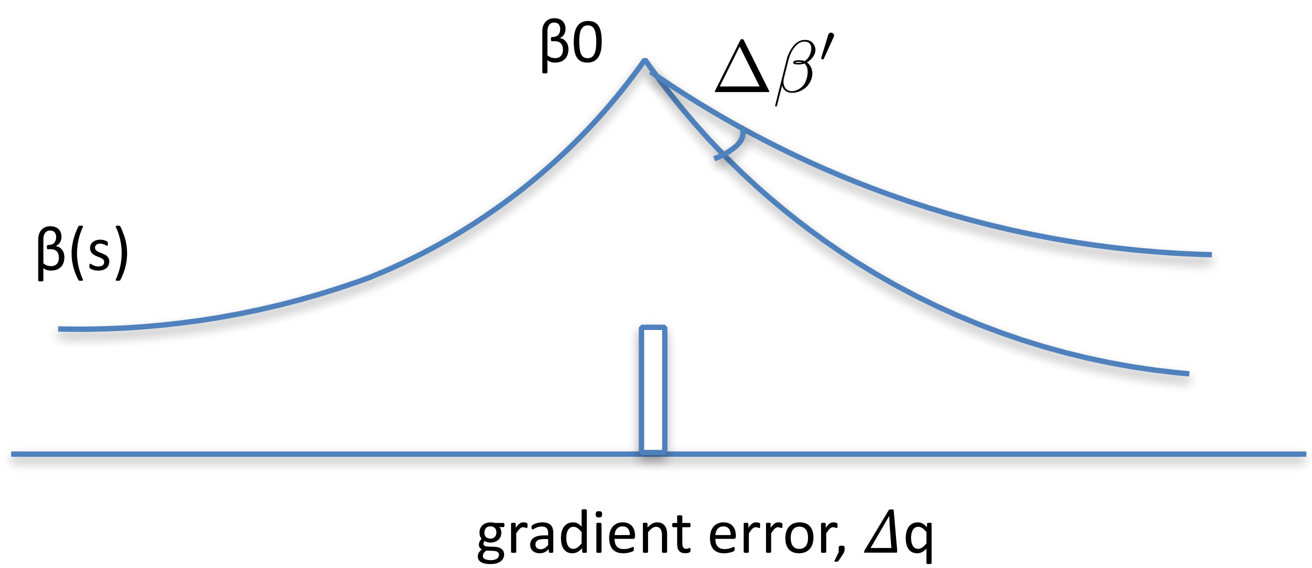
- Sources of gradient focusing errors
  - Quadrupole magnet field error
    - » powering error; control error; manufacturing error
  - Dipole pole tip error (non-parallel poles)
  - etc.

- Impact of gradient errors
  - Look at Hill's Equation: $$x'' + K(s)x = 0$$
    - » errors in the values of K will alter…
      - phase advance (tune, or betatron frequency)
      - amplitude function, β

Focusing (quadrupole) Errors

- β, α distortions and "beta-beat"



$$\Delta\alpha = -\frac{1}{2}\Delta\beta'$$

if ideal gradient produces strength $q = B'\ell/(B\rho)$,
then a gradient error will produce $\Delta q = \Delta B'\ell/(B\rho)$
and the slope of β will change according to

$$\boxed{\Delta\alpha = \beta_0 \Delta q}$$

Downstream the distortion will propagate:

$$\frac{\Delta\beta}{\beta}(s) \approx -\Delta q \beta_0 \sin 2\psi_0(s)$$

β Distortion in a Synchrotron

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- In a circular accelerator, the closed solution of the amplitude function(s) will be altered by the gradient error. With analysis similar to the situation for a closed orbit distortion, the gradient error will produce a closed β-distortion all around the ring according to (for small errors):

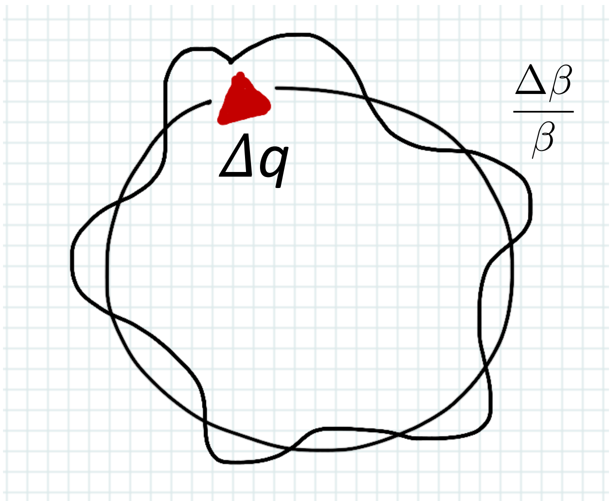
dipole error:

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi$$

quad error:

$$\frac{\Delta\beta}{\beta}(s) \approx -\Delta q\beta_0\sin 2\psi_0(s)$$

$$\frac{\Delta\beta}{\beta}(s) \approx -\frac{\Delta q\beta_0}{2\sin 2\pi\nu}\cos(2|\Delta\psi| - 2\pi\nu)$$



Focusing (quadrupole) Errors

- Phase/tune shift

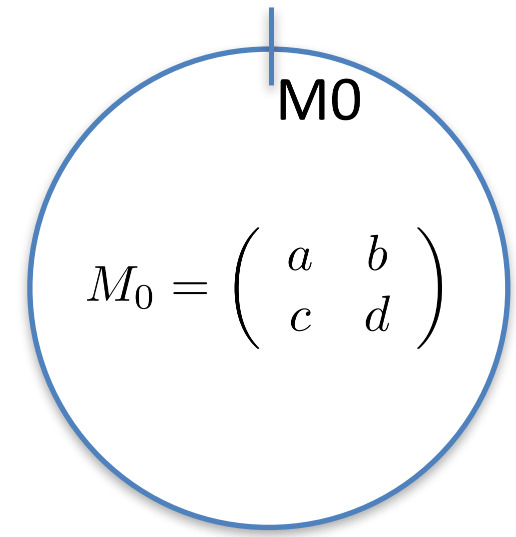
  - a gradient error will distort the amplitude function, and therefore distort the development of the phase advance downstream. As the β distortion will oscillate about the ideal β function, the phase advance will slightly increase and decrease along the way. This is particularly important in a ring where the betatron tune, ν, might need fine control.

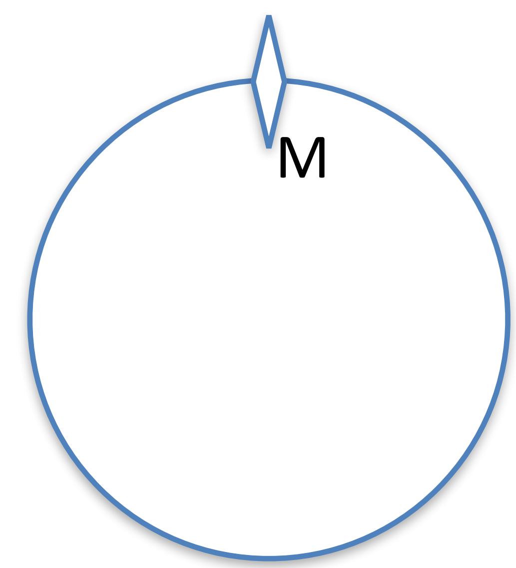
  - To see the change in tune for a synchrotron, we look at the effect on the matrix for one revolution…

The Tune Shift Formula

- M0 is the one-turn matrix of ideal ring
- M is the one-turn matrix of the ideal ring followed by a small gradient error of strength *q*:

$$M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0$$





$$\text{then } M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c - aq & d - bq \end{pmatrix}$$

$$\mathrm{trace} M = 2\cos 2\pi\nu = a + d - bq = \mathrm{trace} M_0 - bq = 2\cos 2\pi\nu_0 - (\beta_0 \sin 2\pi\nu_0) q$$

$$\boxed{\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2} q\beta_0 \sin 2\pi\nu_0}$$

$$\begin{aligned} \cos 2\pi\nu &= \cos 2\pi(\nu_0 + \Delta\nu) \\ &= \cos 2\pi\nu_0 \cos 2\pi\Delta\nu - \sin 2\pi\nu_0 \sin 2\pi\Delta\nu \\ &\approx \cos 2\pi\nu_0 - 2\pi\Delta\nu \sin 2\pi\nu_0 \end{aligned}$$

$$\text{so,} \quad 2\pi\Delta\nu \sin 2\pi\nu_0 \approx \frac{1}{2} q\beta_0 \sin 2\pi\nu$$

$$\boxed{\Delta\nu \approx \frac{1}{4\pi}\beta_0 q}$$

Focusing (quadrupole) Errors

- What happens if the gradient error is too big?

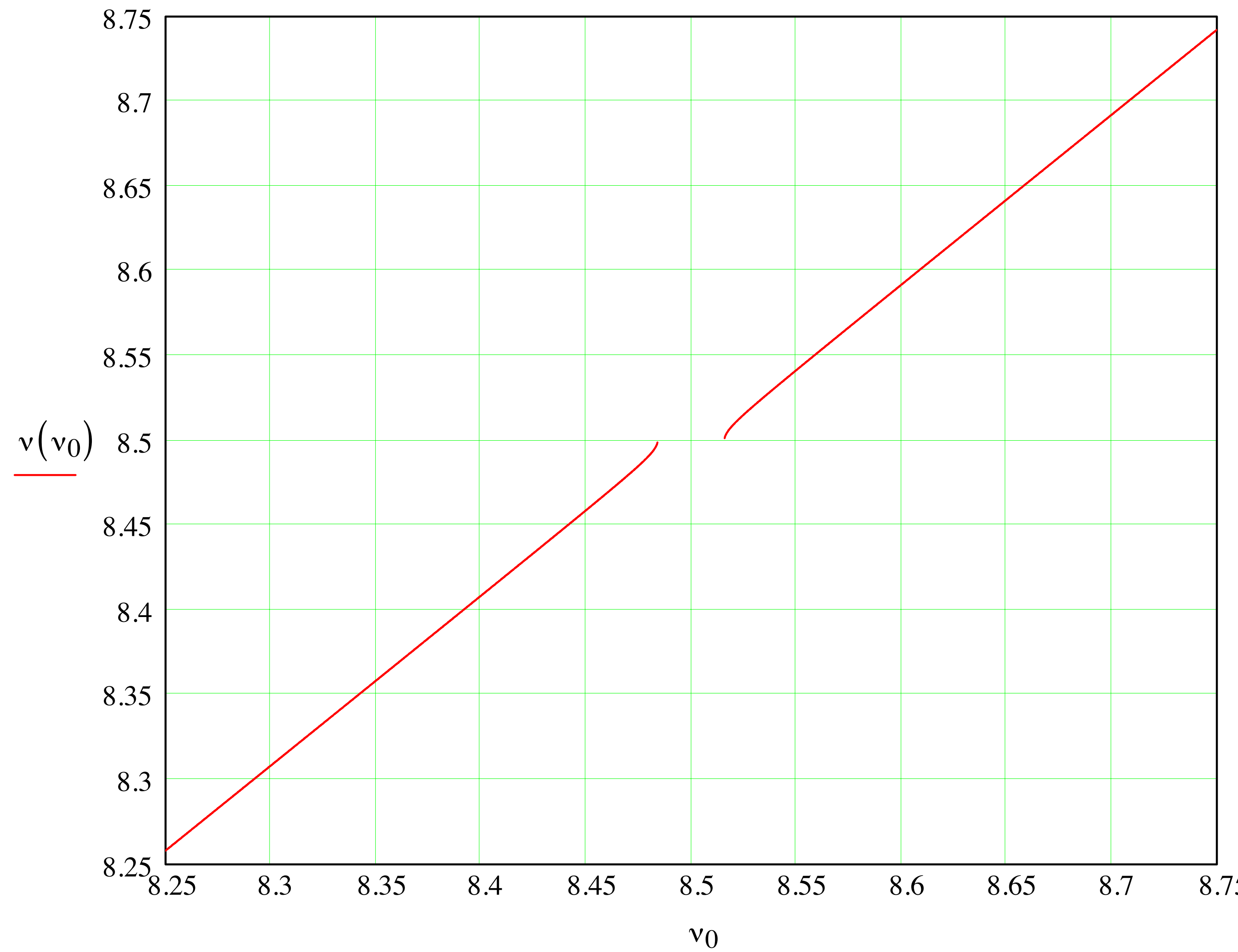
$$\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2} q\beta_0 \sin 2\pi\nu_0$$

if too large, $|\cos 2\pi\nu|$ can become >1, thus unstable!

- Half-integer stop band

$$\frac{\Delta\beta}{\beta}(s) \approx -\frac{\Delta q\beta_0}{2\sin 2\pi\nu}\cos(2|\Delta\psi| - 2\pi\nu)$$

as ν —> integer/2, huge distortions;
a resonance!



Beta-Mismatch Invariant

- We noted that a local gradient error will produce an unintended distortion in the amplitude function (in its slope, in particular):

$$\Delta\alpha = \beta_0 \Delta q$$

- In the absence of further gradient errors,
  - $|\Delta\beta\Delta\gamma - \Delta\alpha^2|$ is an invariant, and thus will have the same value further down the beam line
- proof:

$$J = MJ_0M^{-1} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$J + \Delta J = M(J_0 + \Delta J_0)M^{-1}$$

$$J + \Delta J = MJ_0M^{-1} + M\Delta J_0 M^{-1}$$

$$\Delta J = M\Delta J_0 M^{-1}$$

$$\det(\Delta J) = \det M \det(\Delta J_0) \det M^{-1}$$

$$\det(\Delta J) = \det(\Delta J_0)$$

$$|\Delta\beta\Delta\gamma - \Delta\alpha^2| = invariant$$

Gradient Specifications Discussion

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Tune correction/adjustment

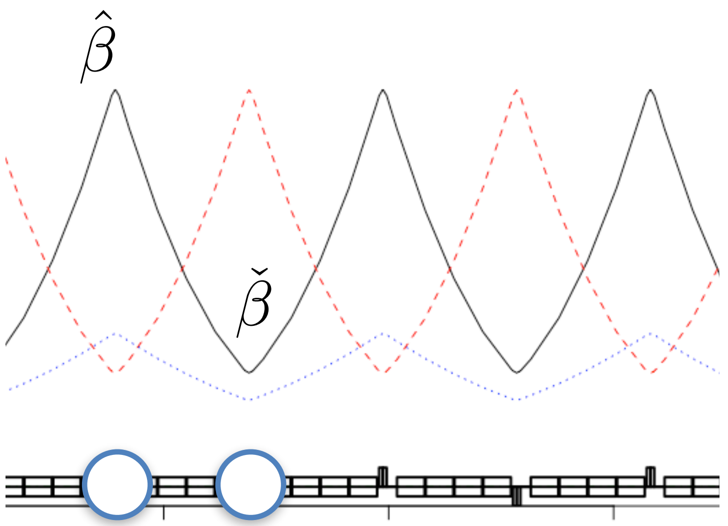
- In the same way that an error will change the tune of a synchrotron, so can a quadrupole field adjustment be made to implement a desired change in the tune
- Note, however, that a quad change will alter the horizontal tune in one direction, but will alter the vertical tune in the other direction. Also, since the amplitude functions, β_x and β_y, may be different, the actual shifts in the two tunes will also be different in magnitude.
- Thus, to exercise independent control of ν_x and ν_y, there needs to be two quadrupoles (or 2 circuits)

Tune correction/adjustment

- Suppose we have a FODO arrangement, and we put adjustable quadrupoles near every "main" quadrupole (N = # quads):

$$\Delta\nu_x = \frac{N}{4\pi}\left[\hat{\beta}\Delta q_1 + \check{\beta}\Delta q_2\right]$$

$$\Delta\nu_y = -\frac{N}{4\pi}\left[\check{\beta}\Delta q_1 + \hat{\beta}\Delta q_2\right]$$



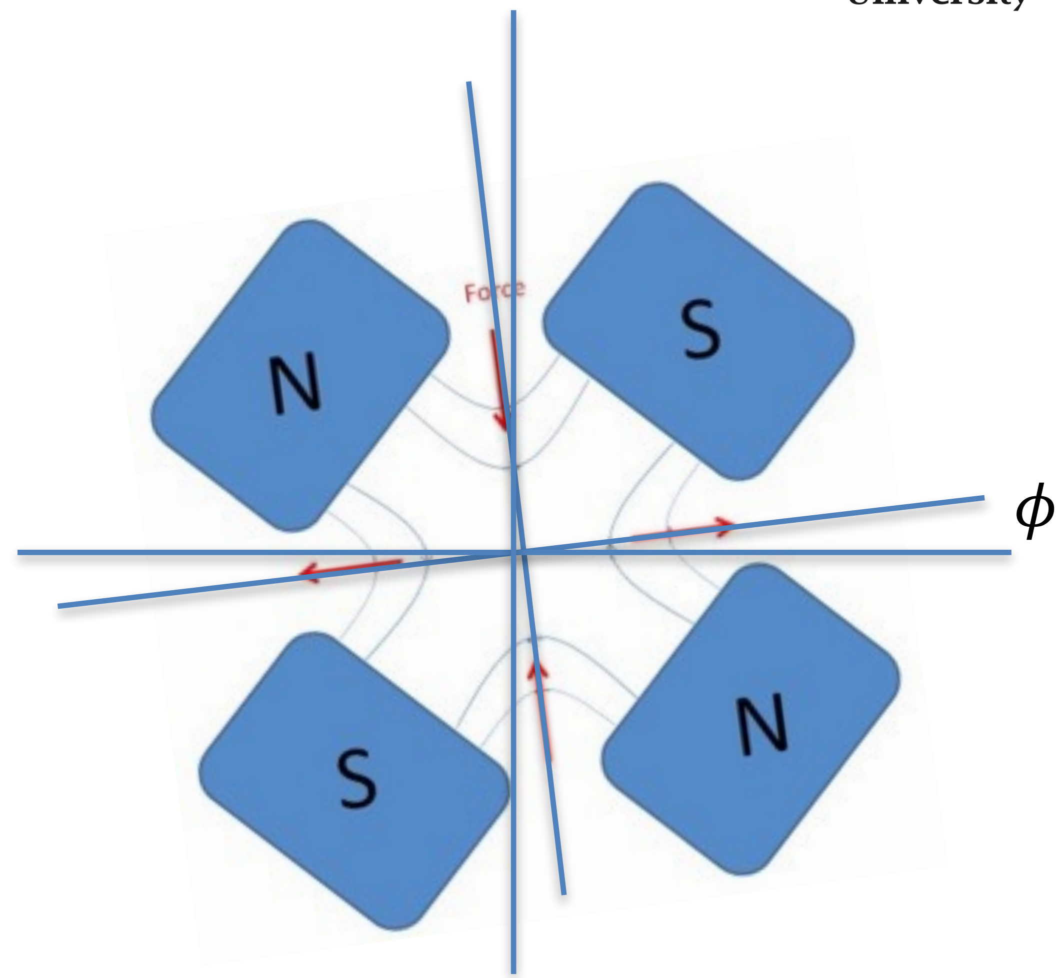
- The quadrupoles can be wired in two separate circuits, and thus the two tunes can be independently adjusted by any (reasonable) amount desired.

Errors creating Linear Coupling

- So far, have discussed systems where the horizontal and vertical motion are distinct. This occurs naturally when using dipole and quadrupole fields:
  - $B_y = B_0 + B' x \qquad B_x = B' y$
  - vertical fields cause motion in x, horizontal fields cause motion in y
- We've seen that a rotated (about its axis) dipole magnet will create a field component in the other plane, causing steering effects. A rotated quadrupole magnet will produce focusing fields that depend on both x and y — coupled motion.

Errors creating Linear Coupling

- Rotated quadrupole magnet



$$
\begin{aligned}
B_x &= B' \cos 2\phi \; x + B' \sin 2\phi \; y \\
B_y &= -B' \sin 2\phi \; x + B' \cos 2\phi \; y
\end{aligned}
$$

a normal quad, rotated by a small angle:

$$
\begin{aligned}
B_y &= B' \; x + 2\phi \; B' \; y \\
B_x &= \underbrace{B' \; y}_{\text{normal}} - \underbrace{2\phi \; B' \; x}_{\text{skew quad}}
\end{aligned}
$$

normal   skew quad, strength: $\frac{\Delta B' \ell}{B\rho} = 2\phi \frac{B' \ell}{B\rho} \equiv k$

Clearly, skew quad field couples the horizontal and vertical motion: $\Delta x' = \frac{B_y \ell}{B\rho} = \frac{\Delta B' \ell}{B\rho} y$

Linear Coupling From Solenoid Fields

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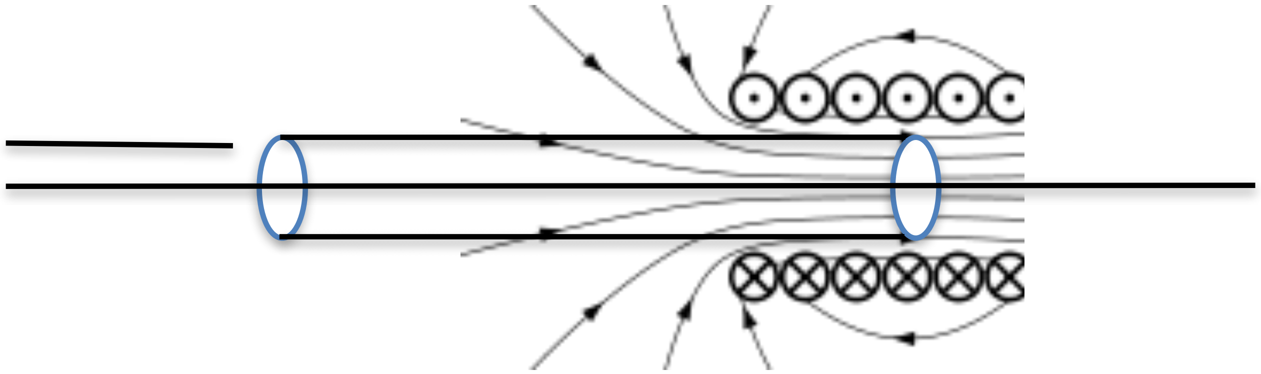
$$\Delta p_\theta \approx q \int_{-\infty}^{0} (\vec{v} \times \vec{B})_\theta dt \qquad = -\frac{qB_0}{2}\, r$$



upon entrance:

$$\begin{aligned} \Delta x' &= \frac{B_0}{2B\rho} y \\ \Delta y' &= -\frac{B_0}{2B\rho} x \end{aligned}$$

(opposite signs upon exit)

through central region:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \rho\sin\theta & 0 & \rho(1-\cos\theta) \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & -\rho(1-\cos\theta) & 1 & \rho\sin\theta \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_0$$

$$\theta = \frac{B_0\ell}{B\rho}$$

$$\rho = \frac{B\rho}{B_0}$$

The motion in each plane depends upon the trajectory in both planes

Effects of coupling on betatron tunes

- Coupling moves the frequencies about — moves the betatron oscillation tunes, in the case of an synchrotron — and so can defeat the precise tune control needed to avoid resonances in devices such as colliders and other storage rings.
- At an even more elementary level, coupling is an irritant in diagnosing beam behavior, for the eigenfrequencies and eigenmodes are no longer associated with the degrees of freedom specified in the design.

Eigen-frequencies of Coupled Oscillator

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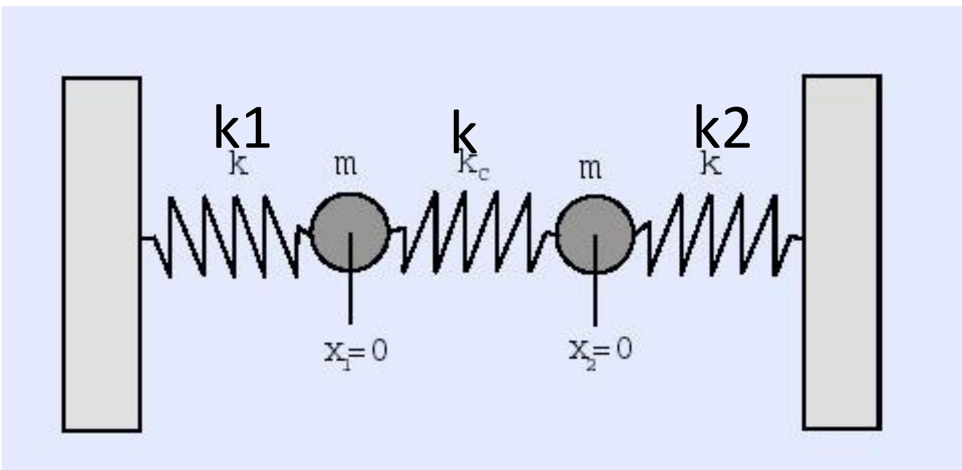
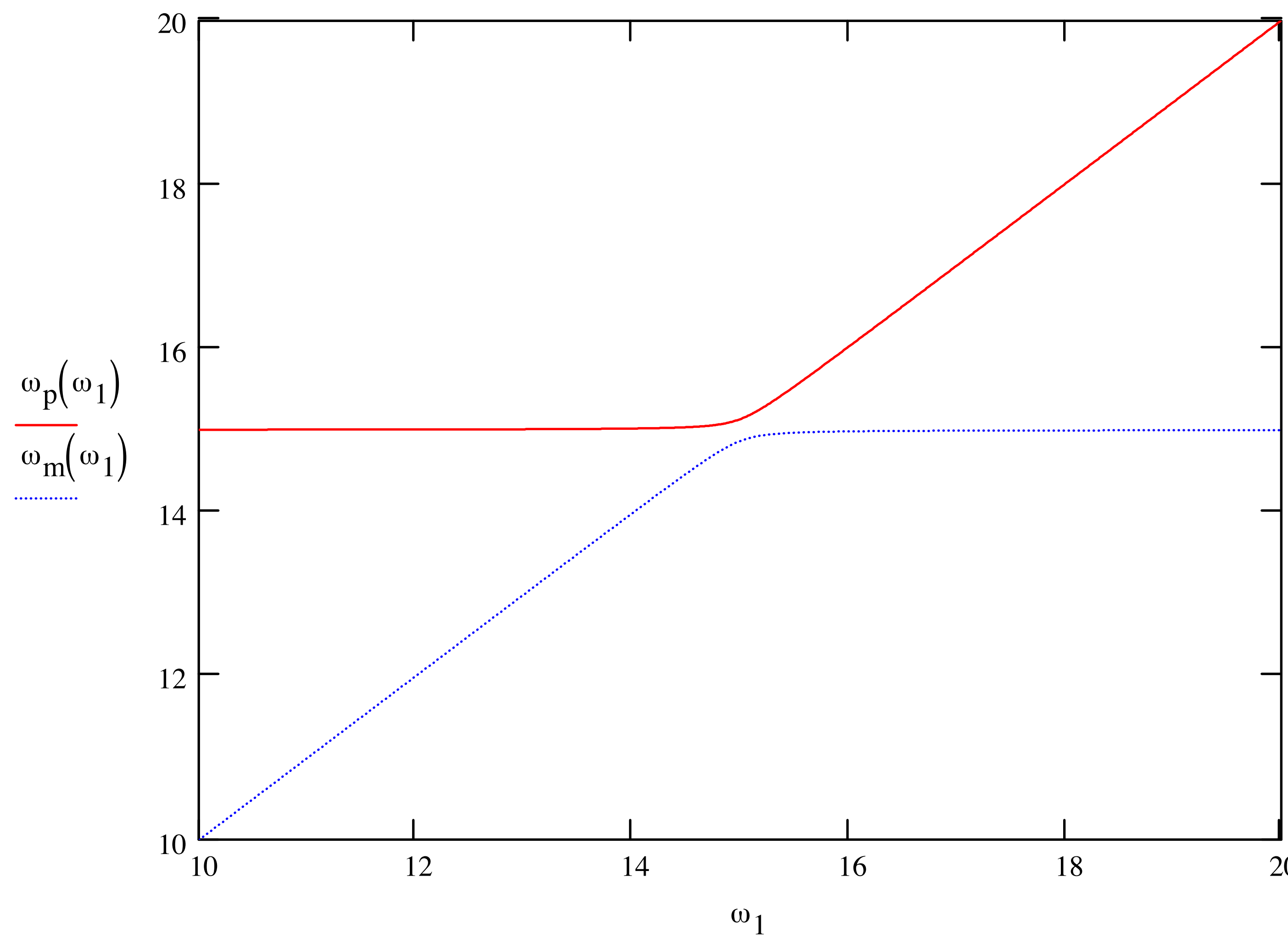


$q := 2$ $\qquad \omega_2 := 15$

$\omega_1 := 10, 10.01 .. 20$

$$\omega_m(\omega_1) := \sqrt{\frac{\omega_1^2 + \omega_2^2 - \sqrt{\left(\omega_2^2 - \omega_1^2\right)^2 + 4 \cdot q^4}}{2}}$$

$$\omega_p(\omega_1) := \sqrt{\frac{\omega_1^2 + \omega_2^2 + \sqrt{\left(\omega_2^2 - \omega_1^2\right)^2 + 4 \cdot q^4}}{2}}$$



In a synchrotron, find that the minimum separation that can be obtained in the presence of a skew quadrupole field is:

$$\Delta\nu_{min} = \frac{|k|}{2\pi}\sqrt{\beta_x\beta_y} = \frac{|\phi q|}{\pi}\sqrt{\beta_x\beta_y}$$

(if due to a rotated quad)

Beam Transport through Coupled Systems

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- We've just seen the possible introduction of a "4x4" matrix approach to analyzing coupled motion
- If we look at 4x4 transport matrices that operate on (x,x',y,y') vectors, then the transport of covariance matrices works just as before:

$$\Sigma = M\Sigma_0 M^T$$

4x4 matrices now

$$\Sigma = \begin{pmatrix} \langle x^2\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle x'x\rangle & \langle x'^2\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \langle yx\rangle & \langle yx'\rangle & \langle y^2\rangle & \langle yy'\rangle \\ \langle y'x\rangle & \langle y'x'\rangle & \langle y'y\rangle & \langle y'^2\rangle \end{pmatrix}$$

can also extend to 6x6, which includes W-t (or z-z' or z-dp/p, or…)

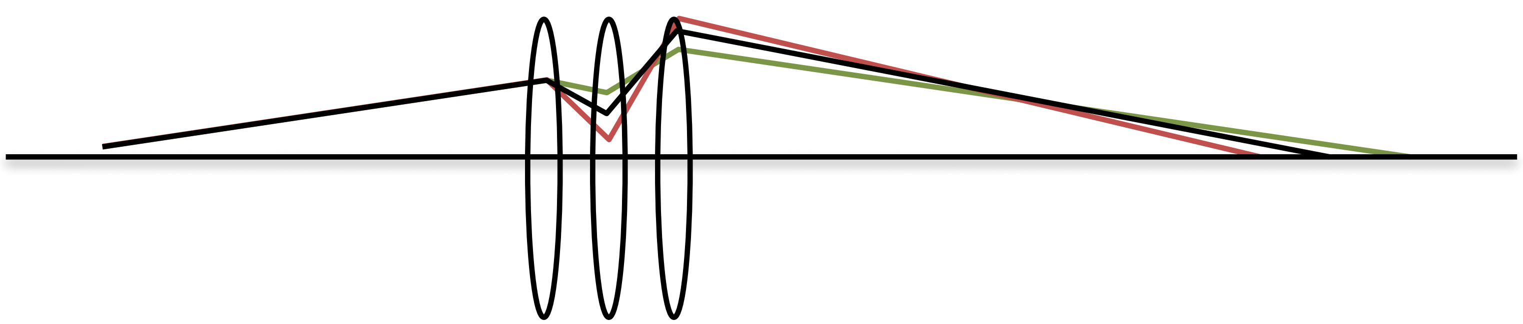
Chromatic Effects

- We may think of dispersion (and the Dispersion function) as being the propagation of a steering error, where the error was introduced due to $\Delta p/p$.
- $\Delta p/p$ will similarly introduce gradient "errors"
  - thus, expect the tune to depend upon $\Delta p/p$
  - and, expect the amplitude function $\beta = \beta(\Delta p/p)$
- Some Examples
  - Chromatic Aberration in a final focus
  - Tune spread in a synchrotron due to momentum — chromaticity

Chromatic Aberration in Final Focus

- Example: *dQ/Q* >> *dp/p*, so dominates the discussion in FRIB design
- In the Final Focus, the beam is big through the final triplet
  - thus, since $\frac{1}{f} = \frac{B'\ell}{B\rho}$
  - we get approximate values at the target:

$$\Delta\alpha = \beta\Delta(1/f) = -(\beta/f)\frac{\Delta p}{p}$$

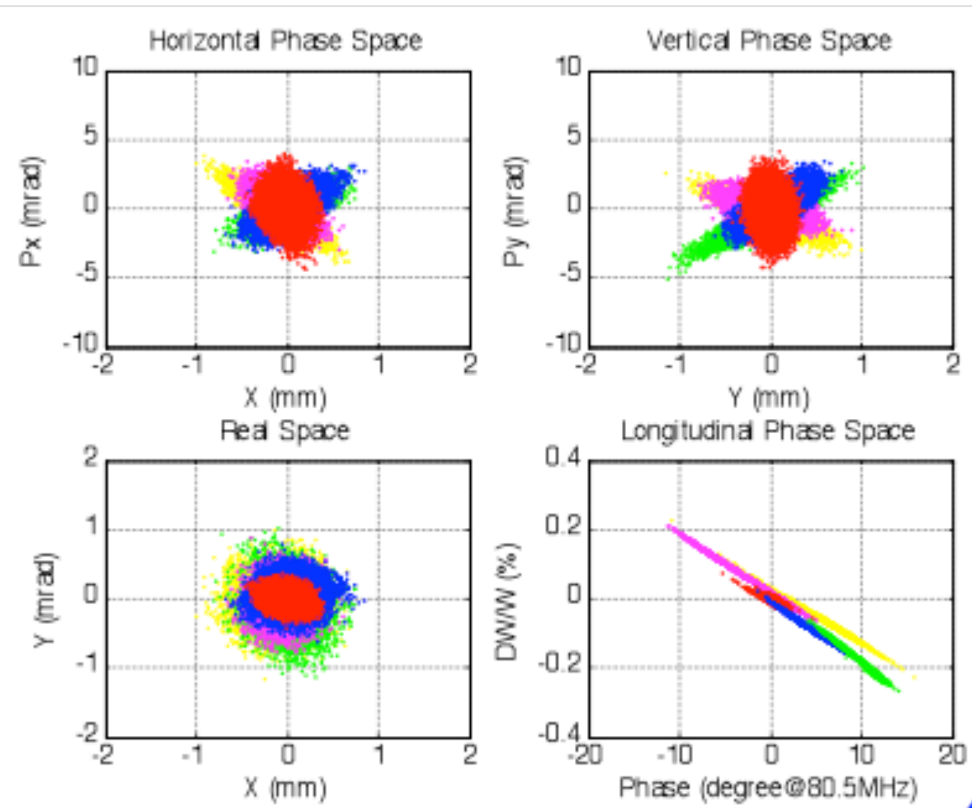


FODO cell: $\frac{\beta}{f} \sim 1 - 5$

Final Focus Triplet: $\frac{\beta}{f} \sim 10^2 - 10^3$

Right: Phase space plots of expected particle distributions at FRIB target. Note the different orientations of the different charge states of the Uranium beam.

$$\Delta(B\rho)/(B\rho) = \Delta p/p - \Delta Q/Q$$



In regions where the amplitude function, β, is large, AND where the quadrupoles are very strong (short f), then the chromatic aberrations become very important

Chromaticity of a Circular Accelerator

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- Chromaticity -- change in the betatron tune, ν, with respect to relative momentum deviation ($\Delta p/p$):

$$x'' + K(s)x = x'' + \frac{qB'(s)}{p}x = 0$$

$$\xi \equiv \frac{\Delta\nu}{\Delta p/p}$$

- There will be a different chromaticity value for each degree of freedom:

$$\xi_x = \frac{\Delta\nu_x}{\Delta p/p}$$
$$\xi_y = \frac{\Delta\nu_y}{\Delta p/p}$$

How to estimate the scale of the effect?

The Natural Chromaticity

- While there may be error fields that contribute to chromatic effects (sextupole fields — later), there will be a "natural" chromaticity due to the ideal magnets of the synchrotron lattice
- Starting from $\Delta\nu = \frac{1}{4\pi}\beta\Delta q$ for a single gradient error,

$$\Delta q \equiv \frac{\Delta B'\ell}{B\rho}$$

$$\Delta\nu = \int \frac{1}{4\pi}\beta(s)\left[-\frac{B'(s)}{B\rho}\frac{\Delta p}{p}\right] ds$$

$$\xi = -\frac{1}{4\pi}\int \beta(s)K(s)ds$$

$$\xi \equiv \frac{\Delta\nu}{\Delta p/p}$$

Can show that for a FODO-style lattice, $\xi \approx -\nu$

Natural Chromaticity of a Low-β Insertion

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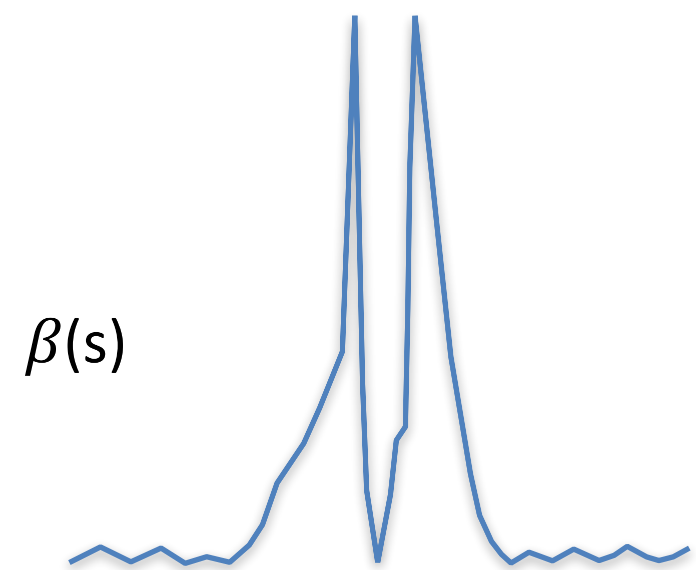
- We saw in our LHC example that the beta function has values:
  - 180 m in cells
  - ~4500 m in final focus triplet
  - 0.5 m at the Interaction Point
- Estimate ξ_{nat} due to IP:



$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

$$\Leftrightarrow \quad K\beta = \gamma + \alpha'$$

$$\int K(s)\beta(s)ds = \int \gamma(s)ds + \int \cancelto{0}{\alpha'(s)}ds$$

$$\int K(s)\beta(s)ds = \int \gamma(s)ds \approx \gamma^* \cdot 2L^* \cdot 2 = 4L^*/\beta^*$$

$$\xi_{IP} \approx -\frac{4L^*}{4\pi\beta^*} = -23\ \mathrm{m}/(\pi\ 0.5\ ) = -15$$

(LHC numbers; note due to FODO cells = -60)

Chromatic Corrections

- Example: suppose synchrotron has ξ = -60, and the beam has a momentum spread of ±0.1%; then the particle distribution will have a spread in tunes between $\nu_0 \pm 0.06\ \nu_0$.
- In order to ensure that all particles have the same tunes (hor/ver), within tolerable levels, need to be able to adjust the overall chromaticity of the ring.
- Desire focusing element with a focusing strength that depends on momentum (linearly, preferably).
- This can be accomplished using sextupole fields in regions with horizontal dispersion.

Chromatic Corrections

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- Sextupole Field:

$$\boxed{\begin{aligned} B_y &= \frac{1}{2}B''(x^2 - y^2) \\ B_x &= B''xy \end{aligned}}$$

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = B''x$$ gradient

So here, if “x” is due to Dispersion: $x = D\dfrac{\Delta p}{p}$

$$\Delta\nu = \frac{1}{4\pi}\beta/f$$

then, $$\frac{1}{f} = \frac{(\partial B_y/\partial x)\ell}{B\rho} = \frac{B''\ell}{B\rho}\cdot D\,\frac{\Delta p}{p}$$

$$\boxed{\Delta\xi = \frac{1}{4\pi}\beta\; D\;\frac{B''\ell}{B\rho}}$$

ℓ = length of the sextupole field

Note: since $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \propto D\cdot\frac{\Delta p}{p}$, provides focusing in one plane,
defocusing in the other plane

Thus, need 2 sextuples (or 2 families of sextuples) for
optimal independent corrections/adjustment of ξx, ξy.

Also Note: introduces (intentionally!) a non-linear field!!

Correction/Adjustment of Chromaticity

- Suppose we have a FODO arrangement, and we put adjustable sextuple magnets near every "main" quadrupole (N = # sextupole magnets):

$$
\begin{aligned}
\Delta\xi_x &= \frac{N}{4\pi}\left[\hat{\beta}\hat{D}\Delta S_1 + \check{\beta}\check{D}\Delta S_2\right] \\
\Delta\xi_y &= -\frac{N}{4\pi}\left[\check{\beta}\hat{D}\Delta S_1 + \hat{\beta}\check{D}\Delta S_2\right]
\end{aligned}
$$

$$S \equiv \frac{B''\ell}{B\rho}$$

- The sextupoles can be wired in two separate circuits, and thus the two chromaticities can be independently adjusted by any (reasonable) amount desired.

The Introduction of a Non-Linear Element

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- For the first time in our discussion, have introduced a “non-linear” transverse magnetic field for explicit use in the accelerator system — sextupoles for chromatic and/or chromaticity correction
- This opens the door to new and interesting phenomena, just as in the nonlinear longitudinal motion:
  - phase space distortions
  - tune variation with amplitude
  - dynamic aperture
  - …

Effect on Phase Space due to Single Sextuple

- Track the trajectory of a particle around an ideal ring, but include the kick from a single sextupole every revolution:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' - Sx^2 \end{pmatrix}_n$$

  - transform to new coordinates: $p \equiv \alpha x + \beta x'$

$$\begin{pmatrix} x \\ p \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} x \\ p - \beta S x^2 \end{pmatrix}_n$$

  - transform again:

$$u \equiv \beta S x, \qquad v \equiv \beta S p$$

$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix}_{n+1} = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} \begin{pmatrix} u \\ v - u^2 \end{pmatrix}_n}$$

- The topology of the phase space here only depends upon the choice of tune, ν. Let's see what happens…

Sextupole Tracking Code Demonstration

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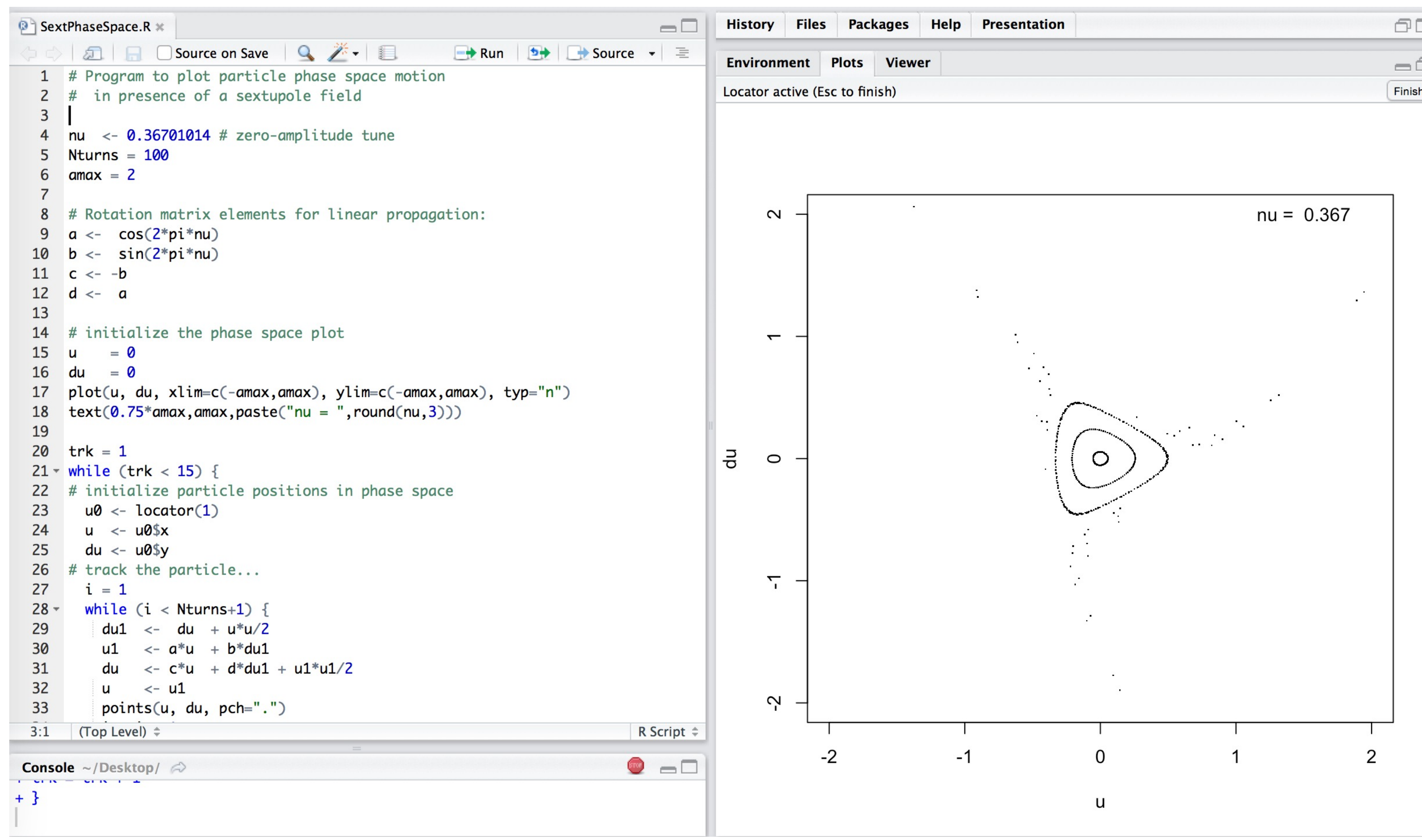
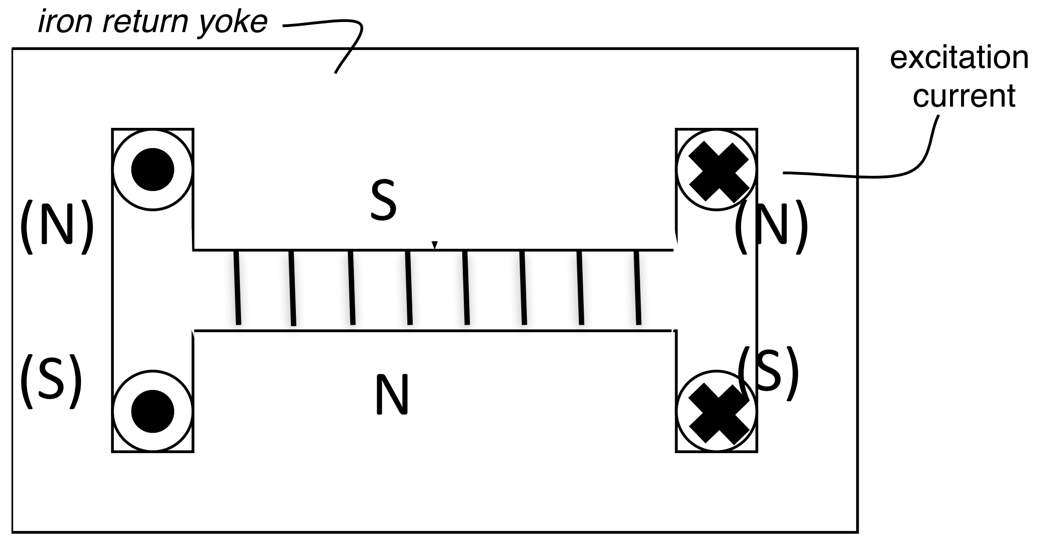
- while (i < Nturns+1) {
- du1 <- du + u*u/2
- u1 <- a*u + b*du1
- du <- c*u + d*du1 + u1*u1/2
- u <- u1
- points(u, du, pch=".")
- i = i + 1
- }



- Let's run a code…

Sources of Transverse Nonlinearities

- Real accelerator magnets
  - Finite width of the field region in a dipole magnet produces a 6-pole (sextupole) term -- By(y = 0) ~ x2
  - Real magnets also have:
    - » Systematic construction errors
    - » Random construction errors
    - » Eddy currents in vacuum
    - » chambers as fields ramp up



- So, real life will introduce sources of line perturbations which can affect the regio

Linear Resonances in Circular Accelerators

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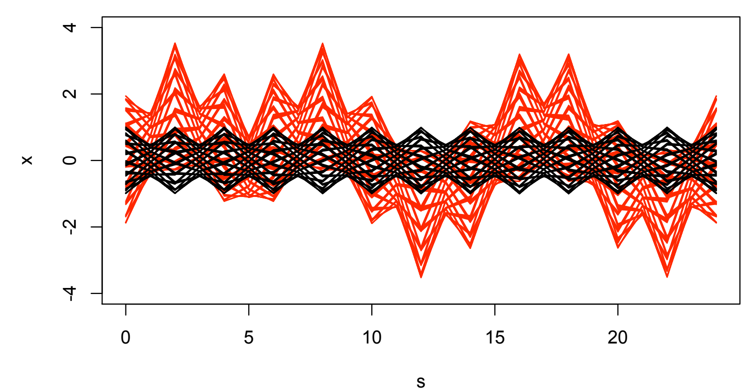
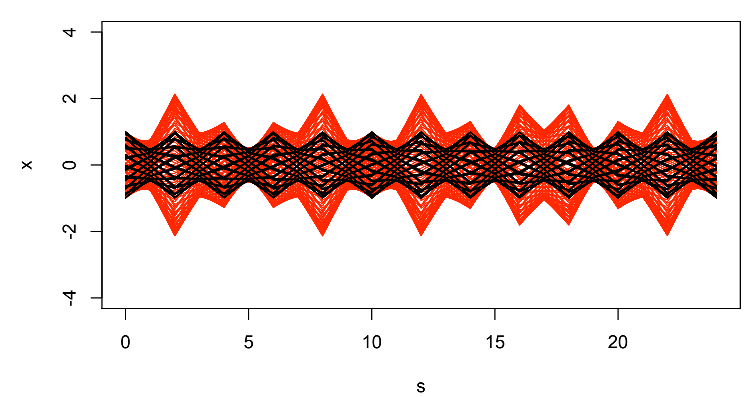
- Imperfections of the ideal “linear elements” lead to implications of the motion

  - guide-field errors
    - » the ‘closed’ trajectory about the synchrotron will become distorted -- average beam trajectory must be adjusted using small, corrector magnets



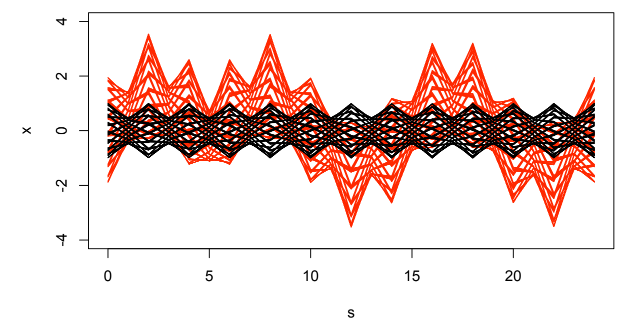
black = ideal
red = distortion

  - focusing field errors
    - » distortions of the beam envelope
    - » if too many, can have |trM| > 2 ==> entire accelerator is unstable



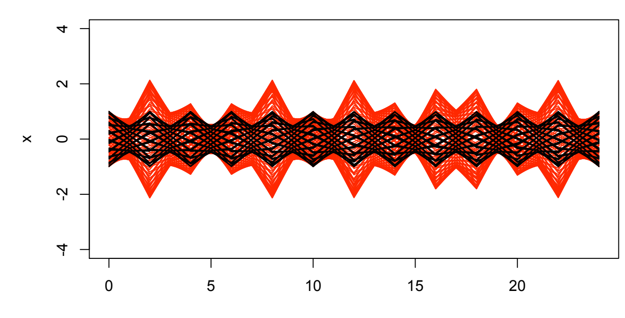
Resonances and Tune Space

- Error fields are encountered repeatedly each revolution -- thus, can be resonant with the transverse oscillation frequency
- Let the "tune" ν = no. of oscillations per revolution
  - repeated encounter with a steering (dipole) error produces an orbit distortion:
    - » thus, avoid integer tunes



$$\Delta x \sim \frac{1}{\sin \pi\nu}$$

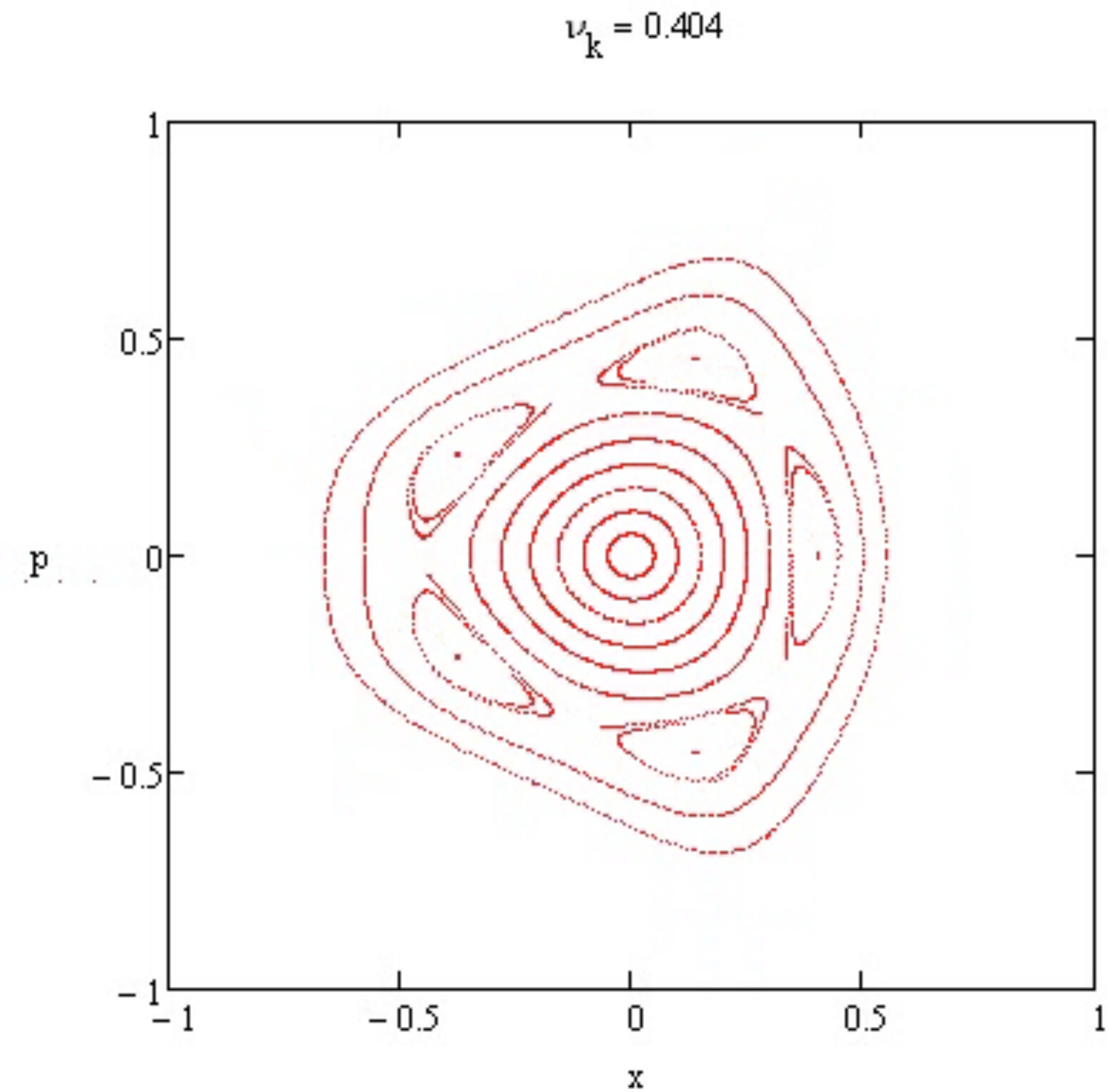
  - repeated encounter with a focusing error produces distortion of amplitude function, β:
    - » thus, avoid half-integer tunes



$$\Delta\beta/\beta \sim \frac{1}{\sin 2\pi\nu}$$

Nonlinear Resonances

- Phase space w/ sextupole field present ($\sim x^2$)
  - topology is tune dependent:
  - frequency depends upon amplitude
  - "dynamic aperture"



With sextupole field present, must avoid tunes:

integer, integer/2, integer/3, ...

"normalized" phase space; ideal trajectories are circular

An Application

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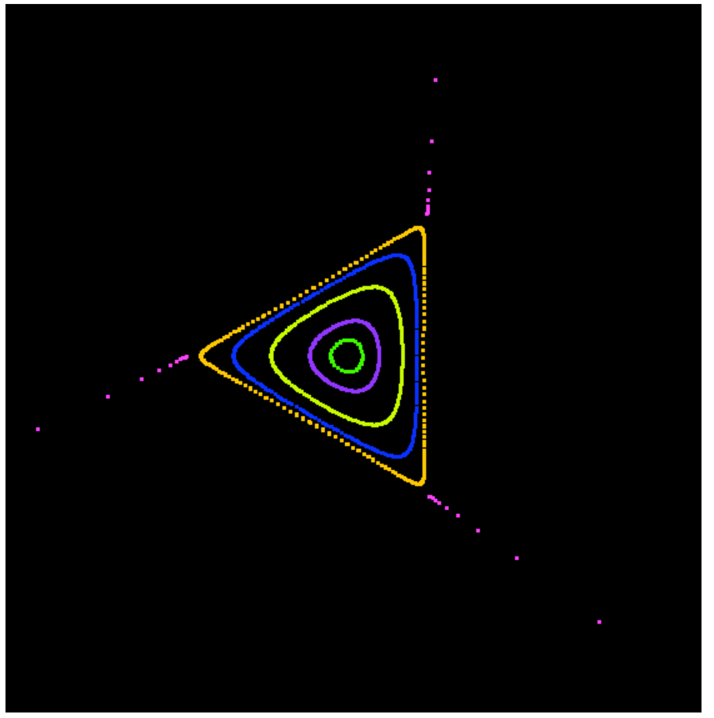
- Put the transverse nonlinear fields to work for us
- Can pulse an electromagnet to send the particles out of the accelerator all at once; but Particle Physics experiments often desire smooth flow of particles from the accelerator toward their detectors
- Resonant Extraction
  - developed in 1960's, particles can be put "on resonance" in a controlled manner and slowly extracted
  - third-integer: carefully approach $\nu = k/3$
    - » driven by sextupole fields
  - half-integer: carefully approach $\nu = k/2$
    - » driven by quadrupole and octupole (8-pole) fields

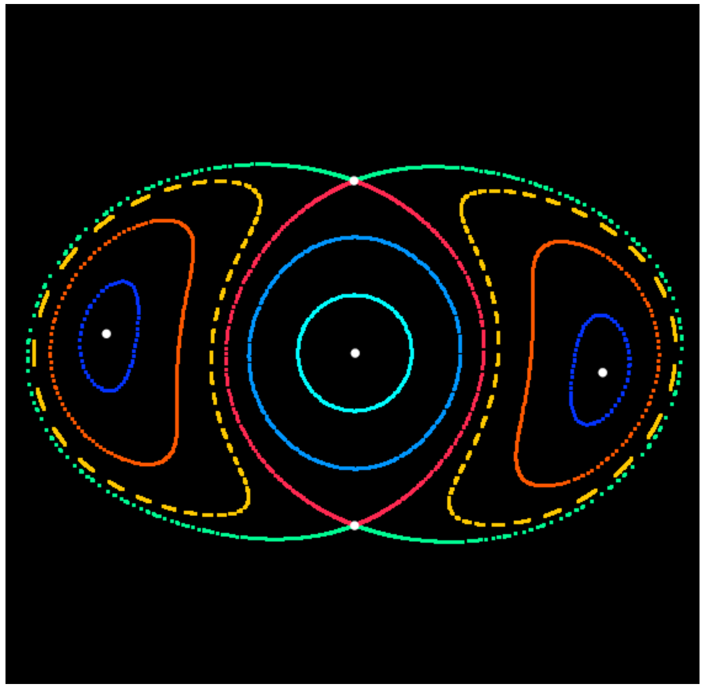
Phase Space used for Extraction

- Linear restoring forces with Sextupole perturbation, running near a tune of k/3



k = “integer”

- Linear restoring forces with Octupole (8-pole) and quadrupole perturbations, running near a tune of k/2



Third-integer Extraction

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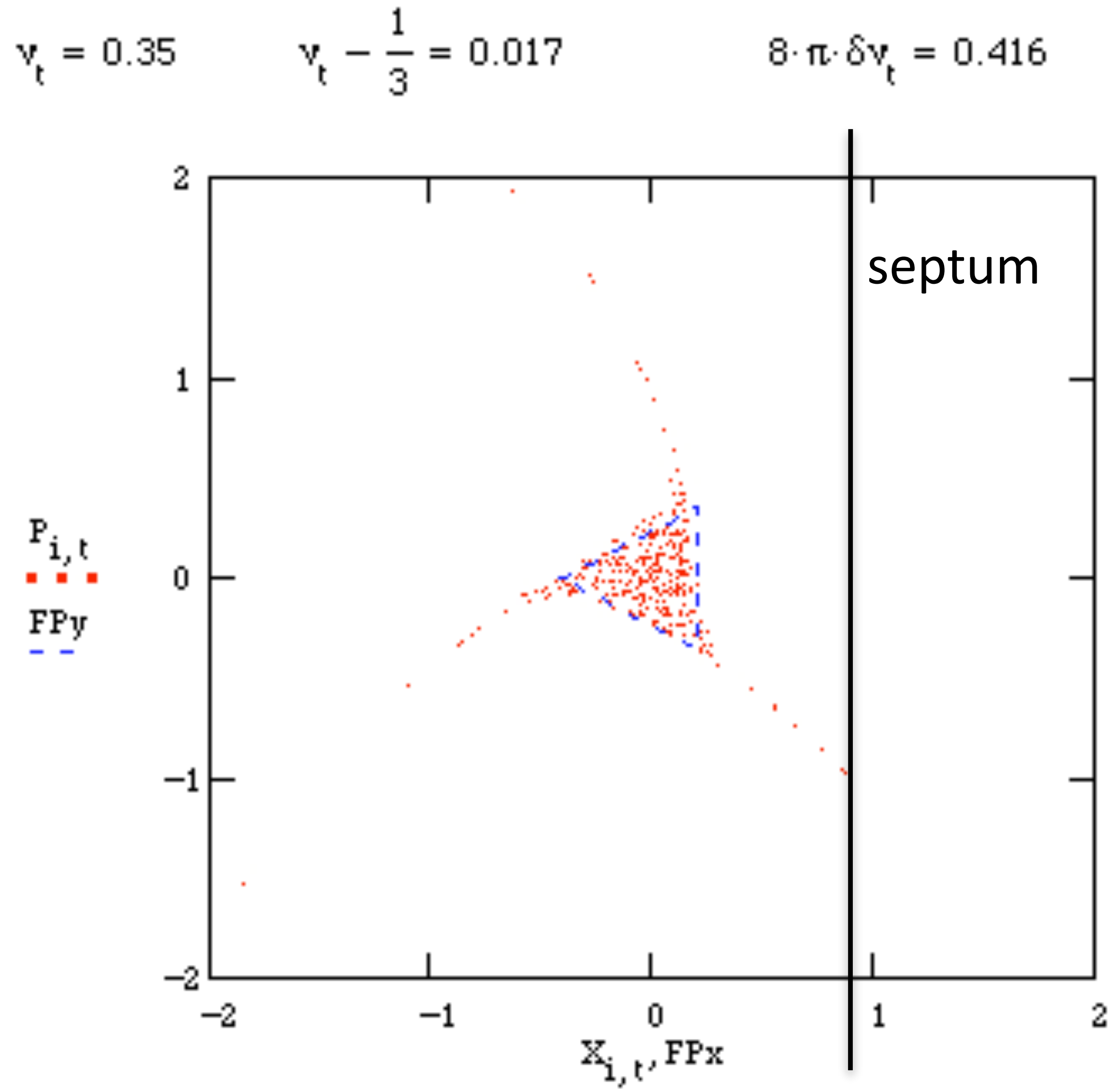
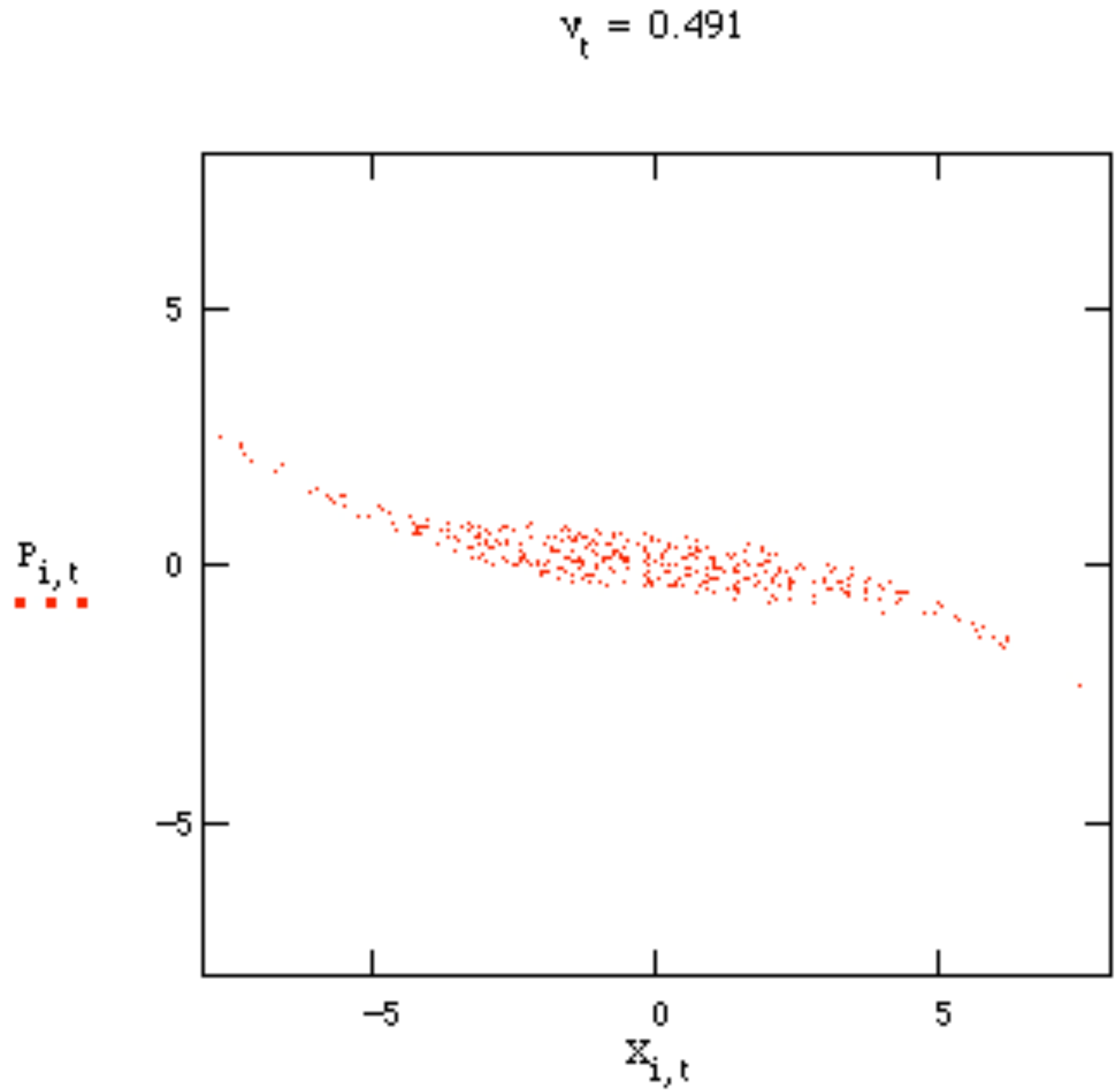
- Example: particles oscillate in phase space in presence of a single sextupole
- Slowly adjust the tune toward a value of k/3
  - (here, k=1)
- Tune is exactly 1/3 at the separatrix
- The lines that appear are derived from a first-order perturbation calculation
- Particles stream away from the "unstable fixed points", stepping across a "septum" which leads out of the accelerator



Half-integer Extraction

- Similar to last movie, but "ideal" accelerator has extra quadrupole and octupole (8-pole) fields
- Slowly adjust the tune toward a value of k/2
  - (here, k=1)
- Here, lowest-order separatrices defined by two intersecting circles
- Eventually, when very close to half-integer tune, entire phase space becomes unstable (|trM|>2)



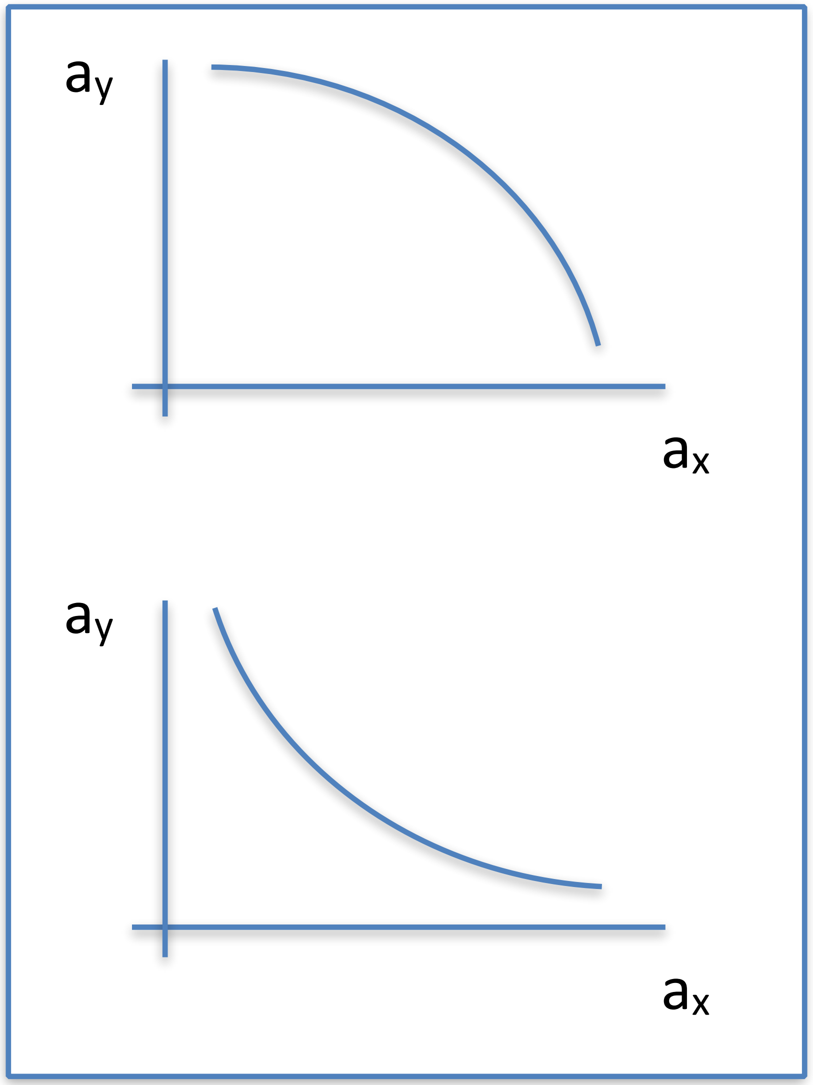
Coupling Resonances

- We've seen that coupling produces conditions where the motion in one plane (x) can depend upon the motion in the other plane (y) and vice versa. When the frequencies of the coupled motion create integer relationships, then coupling resonances can occur:

$$m\,\nu_x \pm n\,\nu_y = k$$

In general, a "difference" resonance will simply exchange the energy between the two planes, back and forth, but the motion remains bounded

A "sum" resonance will exchange energy, but the overall motion can become unbounded



Coupling Resonances

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- Always “error fields” in the real accelerator

- “Skew” fields can couple the motion between the two transverse degrees of freedom

  - thus, can also generate coupling resonances
    - (sum/difference resonances)

  - in general, should avoid: $$m\,\nu_x \pm n\,\nu_y = k$$

avoid ALL rational tunes???

Tune Diagram

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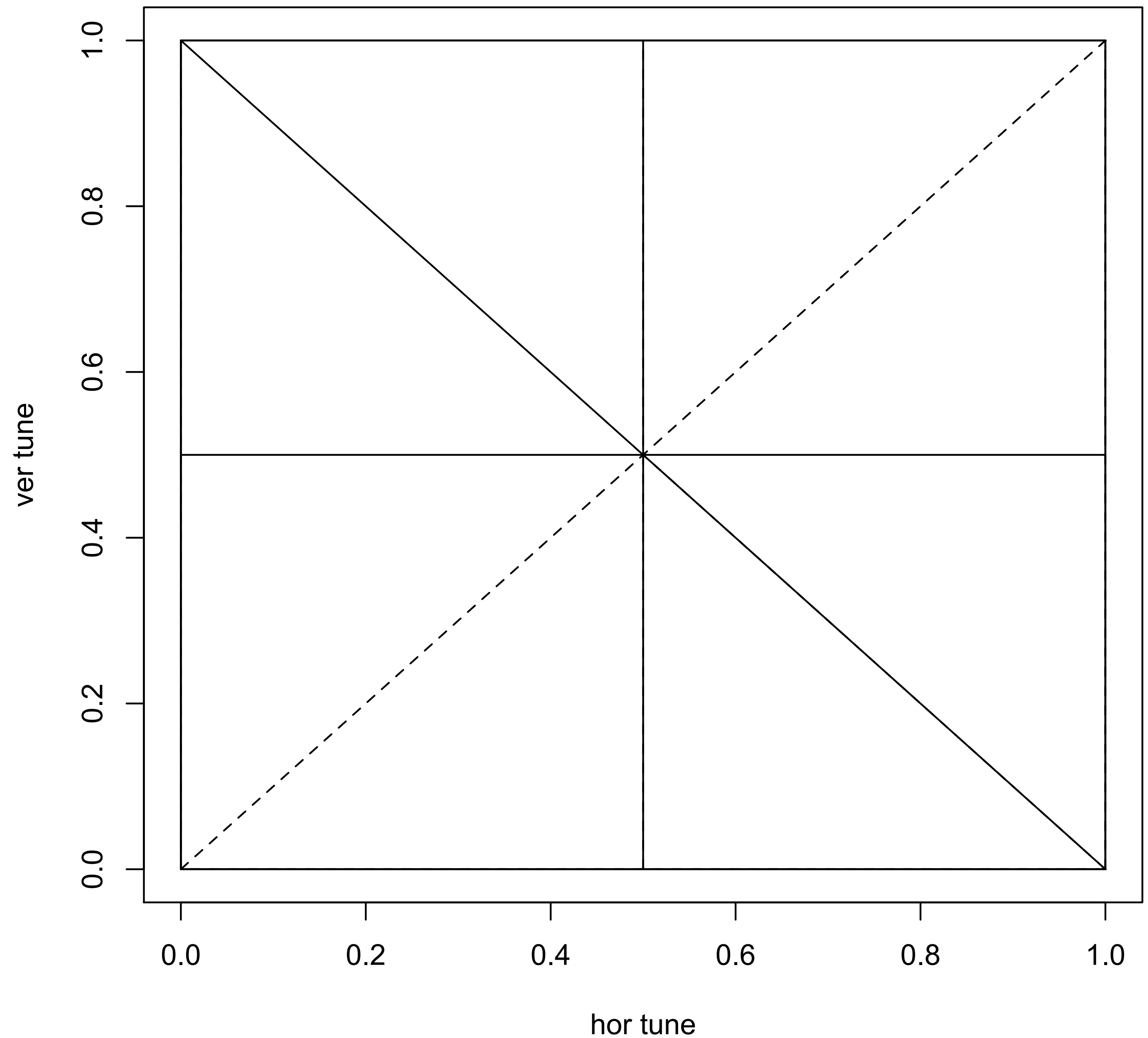
lines of $m\,\nu_x \pm n\,\nu_y = k$

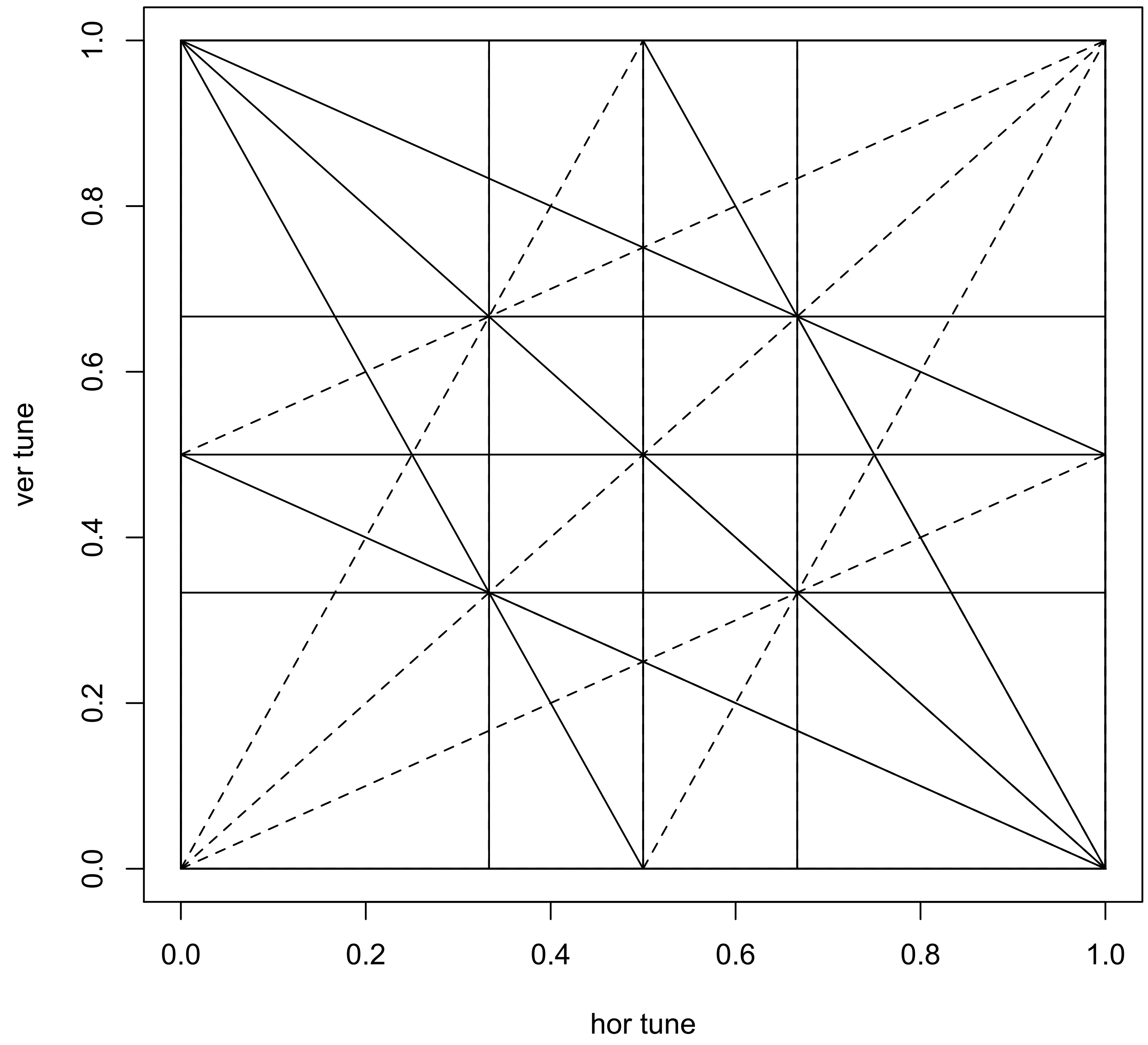
Through order
k =2



Tune Diagram

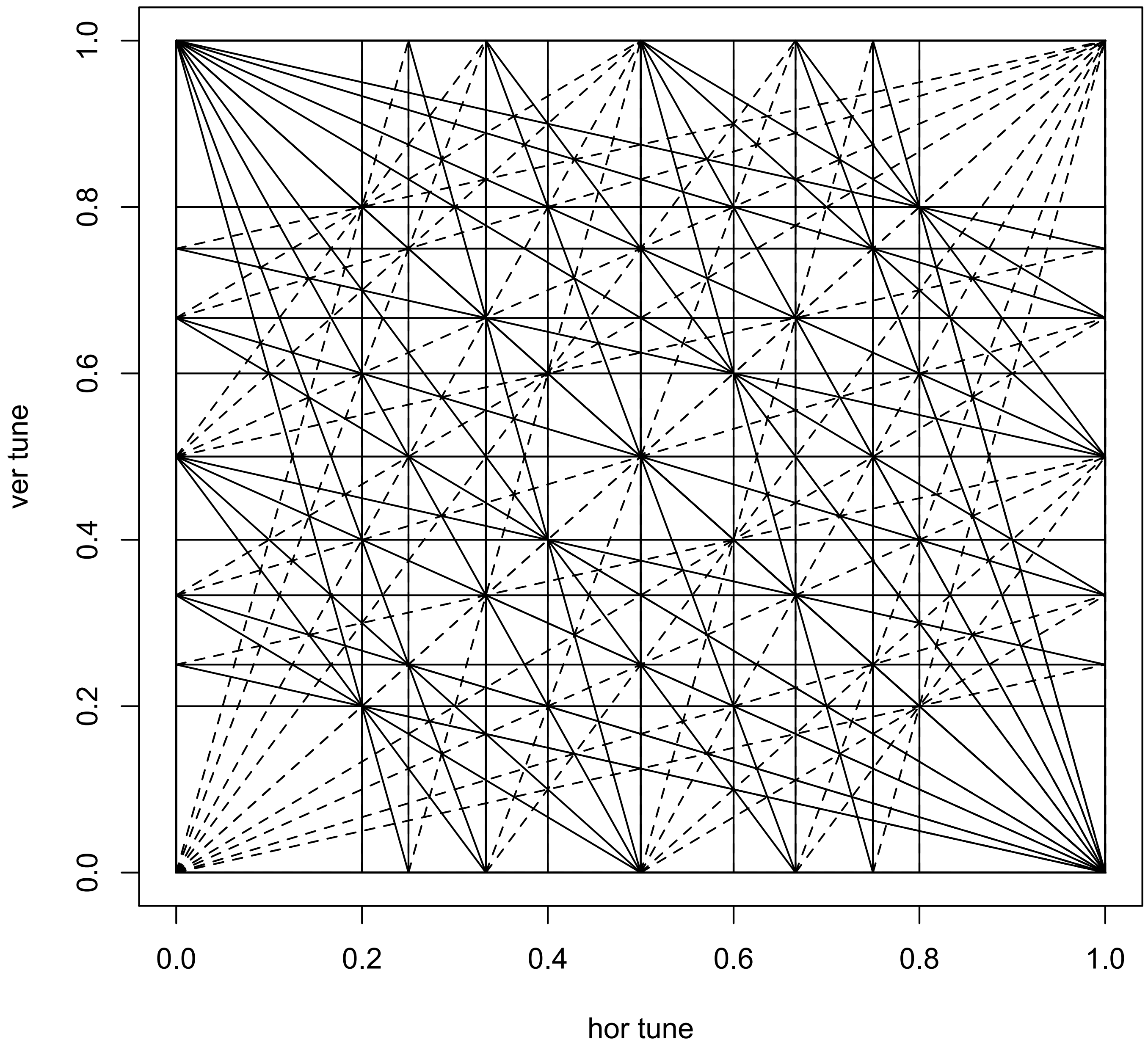
lines of $m\,\nu_x \pm n\,\nu_y = k$

Through order
k =3



Tune Diagram

lines of $m\,\nu_x \pm n\,\nu_y = k$



Through order
k =5

Tune Diagram

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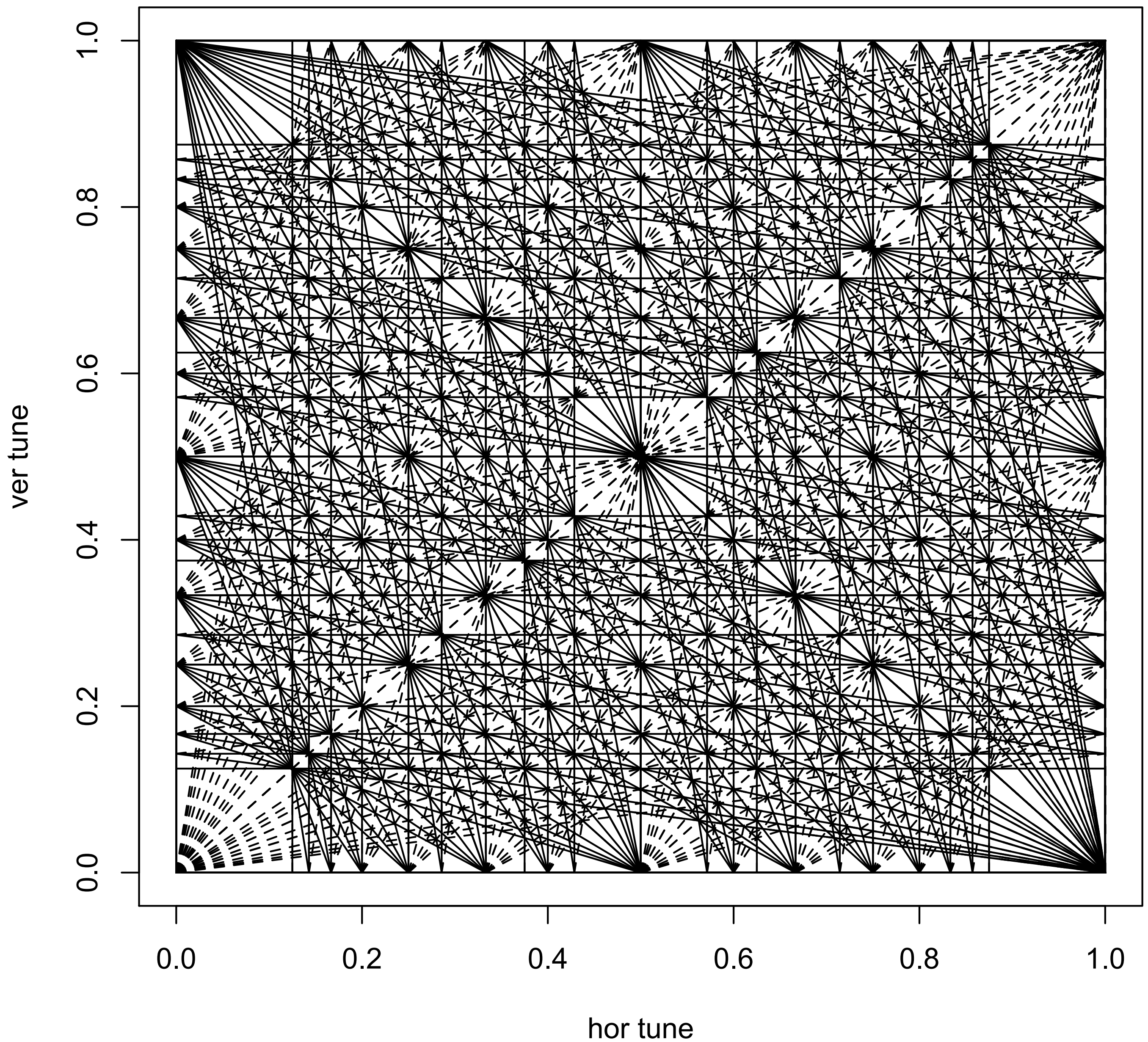
lines of $m\,\nu_x \pm n\,\nu_y = k$



Through order k =8

Tune Diagram

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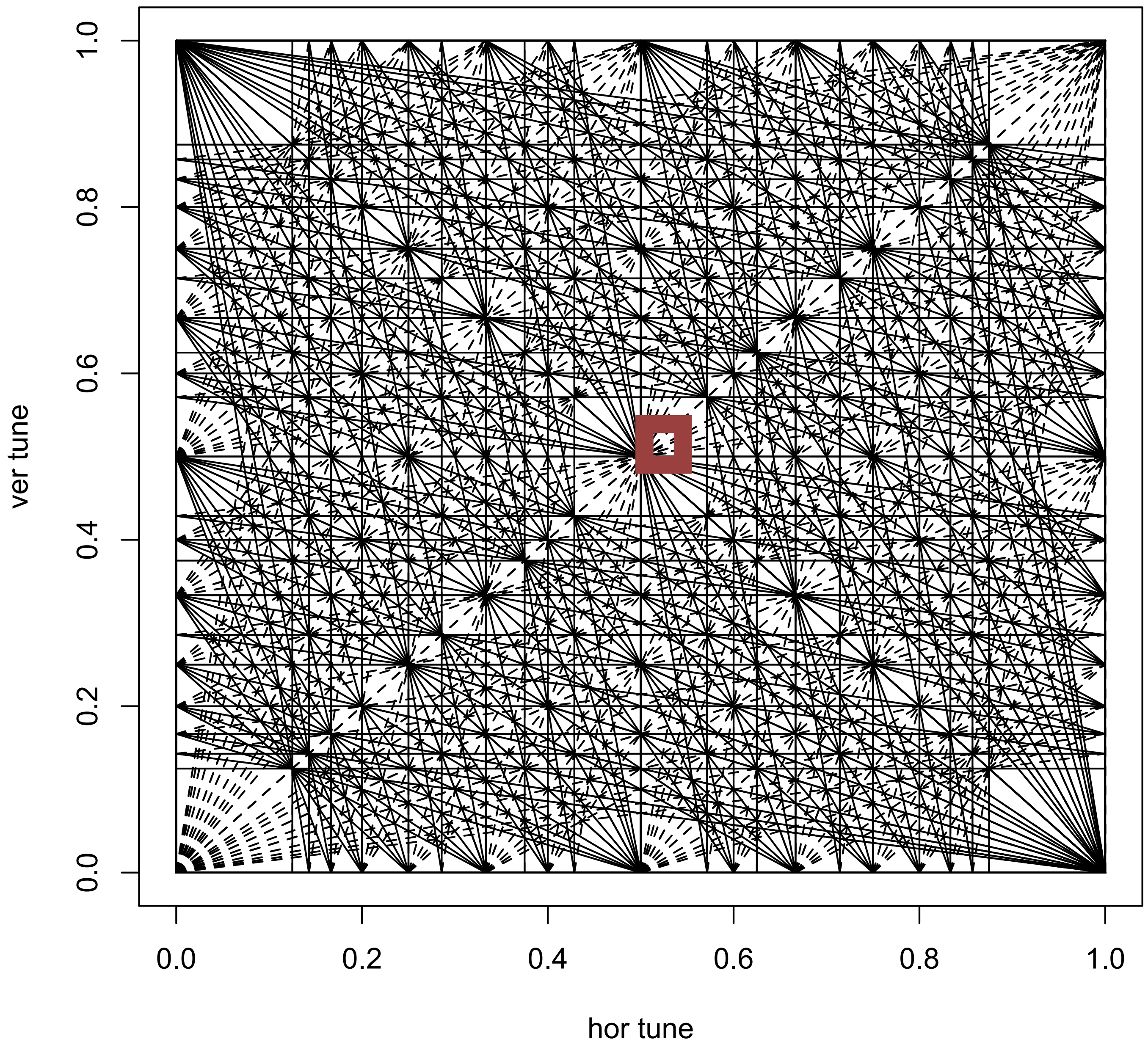
lines of $m\,\nu_x \pm n\,\nu_y = k$



Tune Diagram

lines of $m\,\nu_x \pm n\,\nu_y = k$