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Part II

Analytical Solution of Betatron Motion

Weak Focusing Synchrotron/Betatron

Courant-Snyder Invariant

The Notion of an Amplitude Function…

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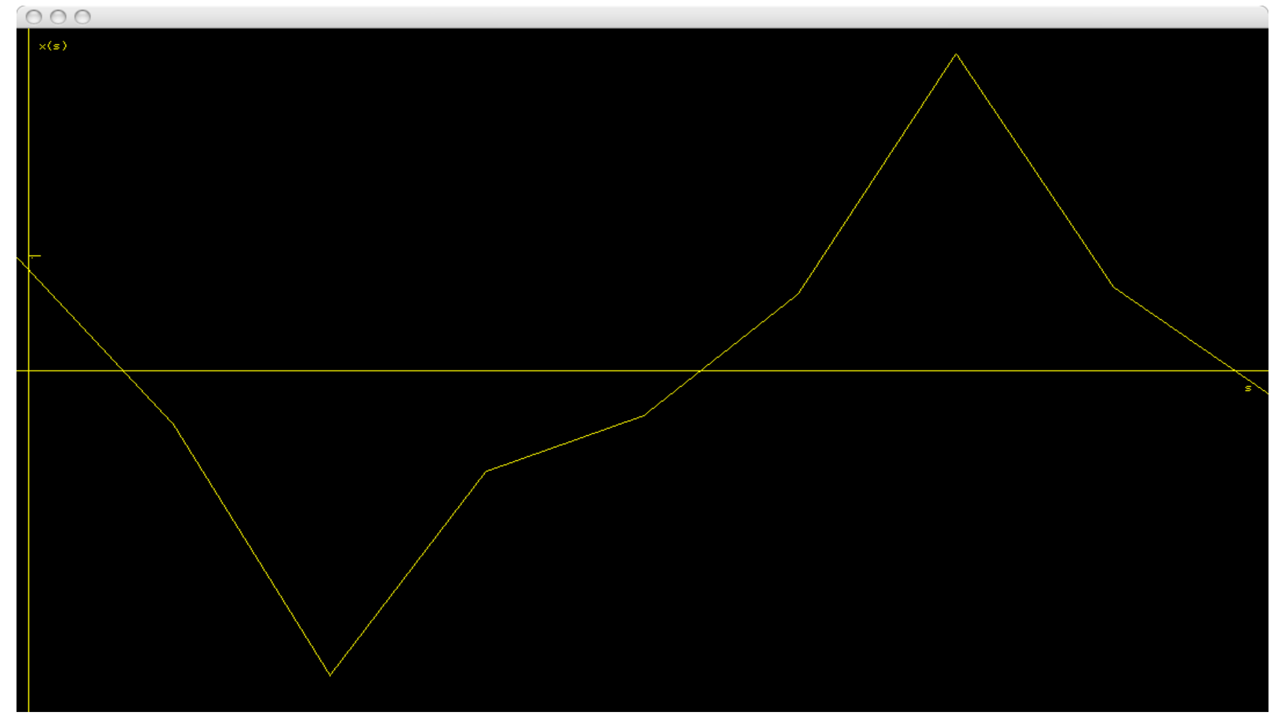
- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line



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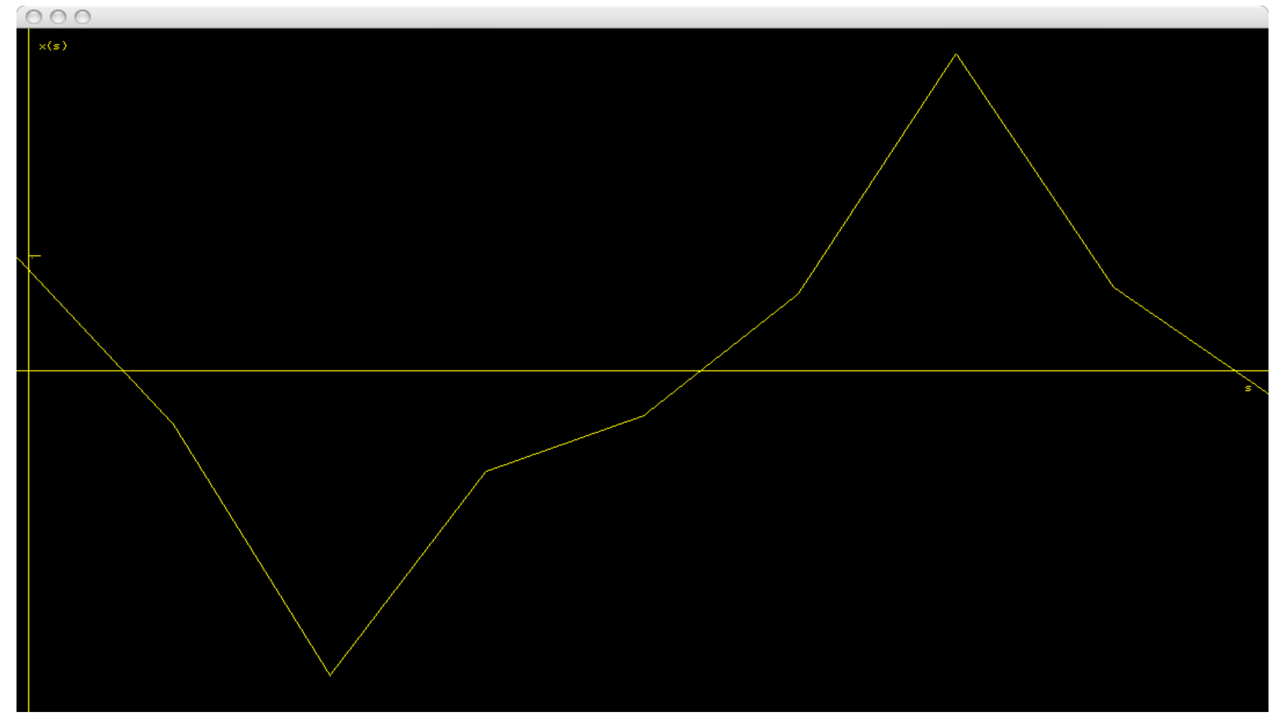

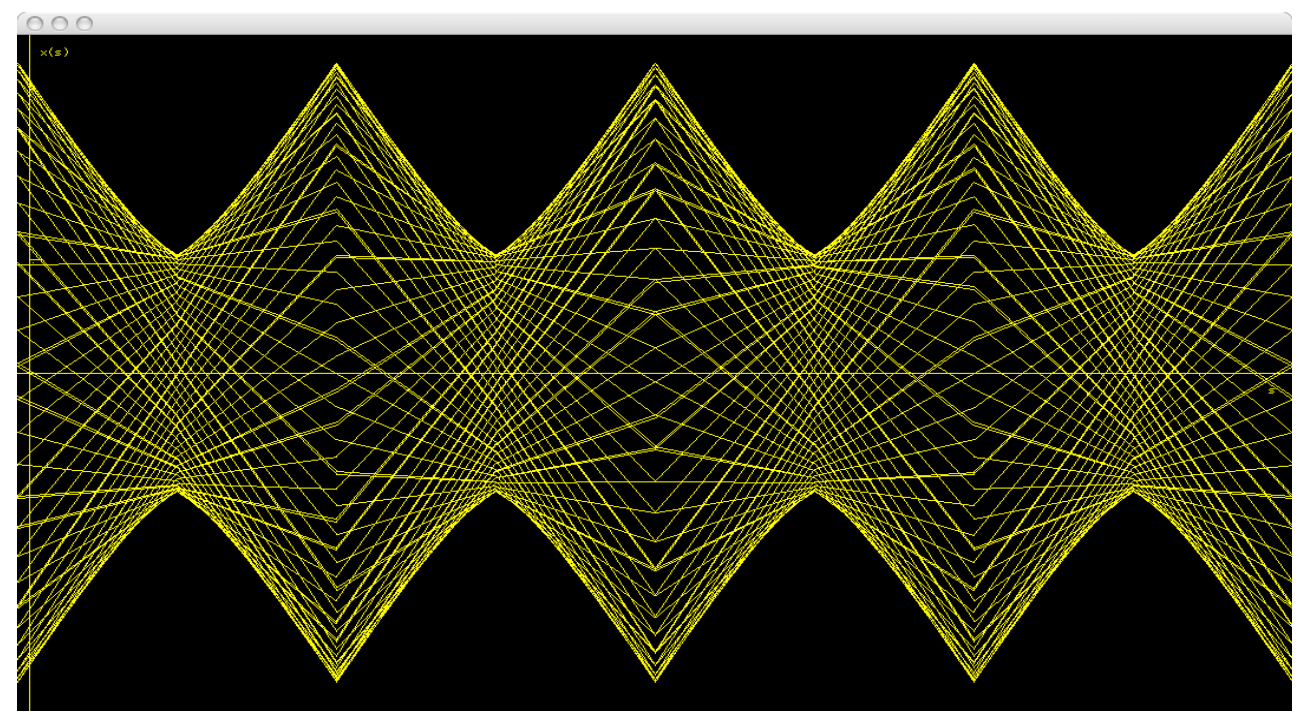

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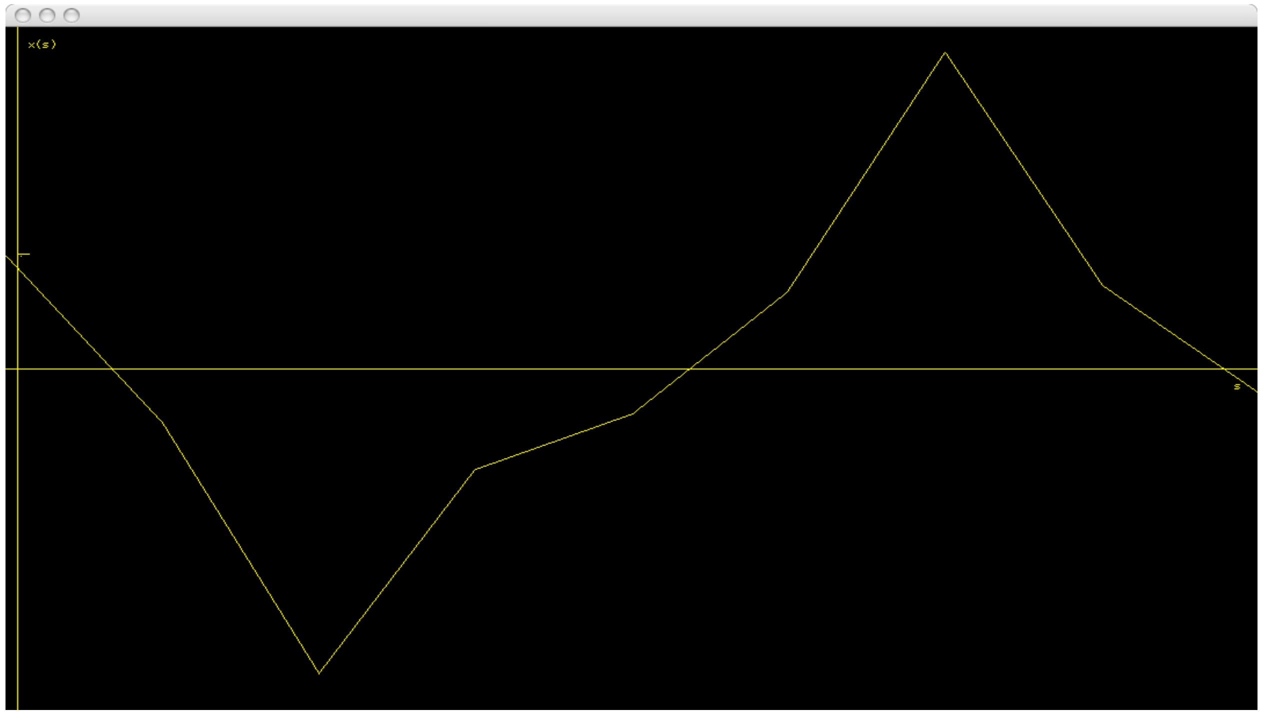

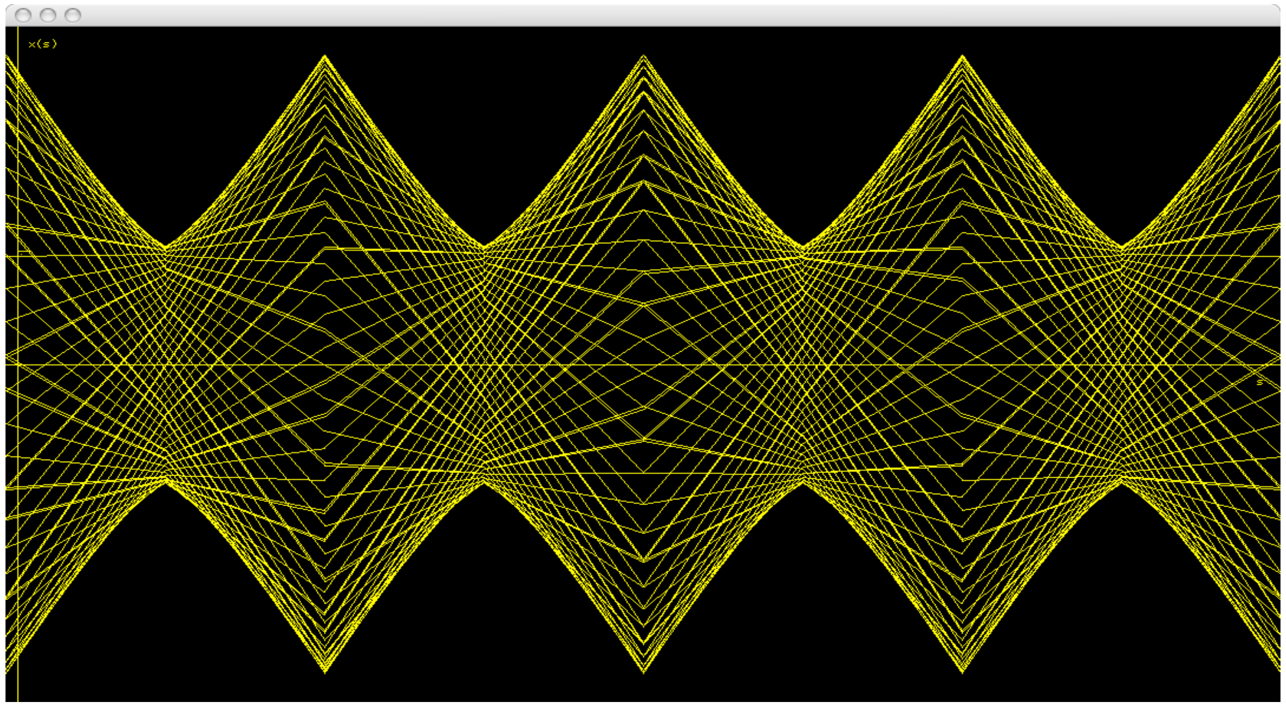

Can we describe the maximum amplitude of particle excursions in analytical form?

of course! *coming up next …*

Pushing the "Envelope"

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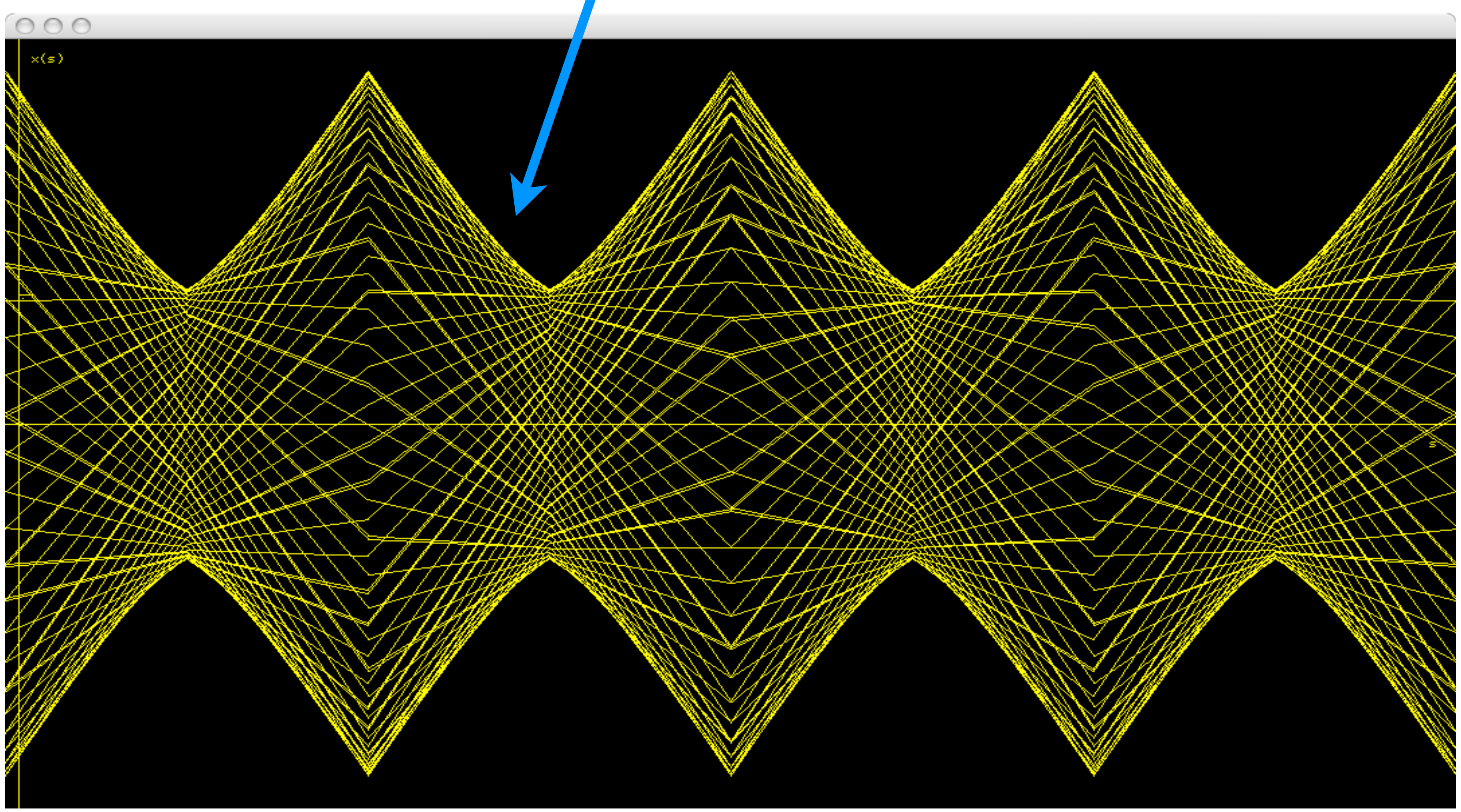
- Wish to look for a functional form of the outer envelope of particle motion, and the rate at which the phase of the oscillatory motion develops within that envelope

- This will enable us to decouple the motion of individual particle from intrinsic properties of the accelerator design

Envelope described by an "amplitude function"



Hill's Equation — Analytical Solution

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- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, *s*.
- So, assume solution is sinusoidal, with a phase which advances as a function of location *s*; also assume amplitude is modulated by a function which also depends upon *s*:
- Then, plug into Hill's Equation ...

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

Analytical Solution (cont'd)

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$$
\begin{aligned}
x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\
x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\
x'' &= \ldots
\end{aligned}
$$

Plugging into Hill's Equation, and collecting terms...

$$
\begin{aligned}
x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\
&\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0
\end{aligned}
$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, must have contents of each [ ] = 0 simultaneously for *sum* = 0.

Analytical Solution (cont'd)

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s)+\delta]$$

- Thus, we must have ...

thus, we need

$$\psi'' + \frac{\beta'}{\beta}\psi' = 0$$

and

$$-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0$$

Analytical Solution (cont'd)

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$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Note: the phase advance is an observable quantity. So, while could choose different value of `const`, then β would just scale accordingly; thus, valid to choose `const` = 1.

Analytical Solution (cont'd)

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$$\beta\psi'' + \beta'\psi' = 0$$

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$$\psi' = 1/\beta$$

$$2\beta\beta'' - (\beta')^2 - 4\beta^2(\psi')^2 + 4K\beta^2 = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

Differential equation that the amplitude function must obey

The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Note: the phase advance is an observable quantity. So, while could choose different value of *const*, then β would just scale accordingly; thus, valid to choose *const* = 1.

Some Comments

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But the amplitude function is also a local wavelength of the motion.
- This seems strange at first, but ...
  - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
  - Thus, the spacing and/or strengths (i.e., *K(s)*) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes must be tied together.

Equation of Motion of Amplitude Function

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From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\boxed{\beta'' + 4K\beta = const.}$$

is the general equation of motion for the amplitude function, β.

(in regions where K is either zero or constant)

Piecewise Solutions

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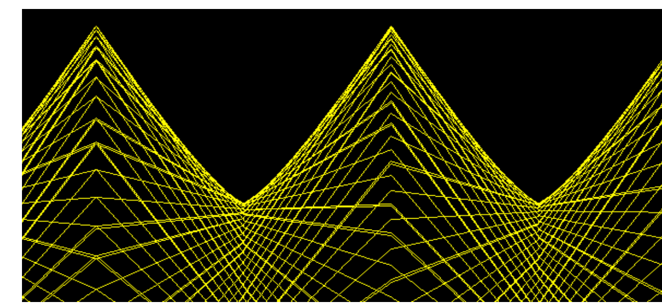
- $K = 0$: $$\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$ Parabola!
  - since $\beta > 0$, then from original diff. eq. …

$$2\beta\beta'' - (\beta')^2 = 4 \qquad \beta'' = \frac{4 + (\beta')^2}{2\beta} > 0$$

  - Therefore, parabola is ***always*** concave up



- $K > 0$~~, $K < 0$~~: sinusoidal + *constant*

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$$

Summary

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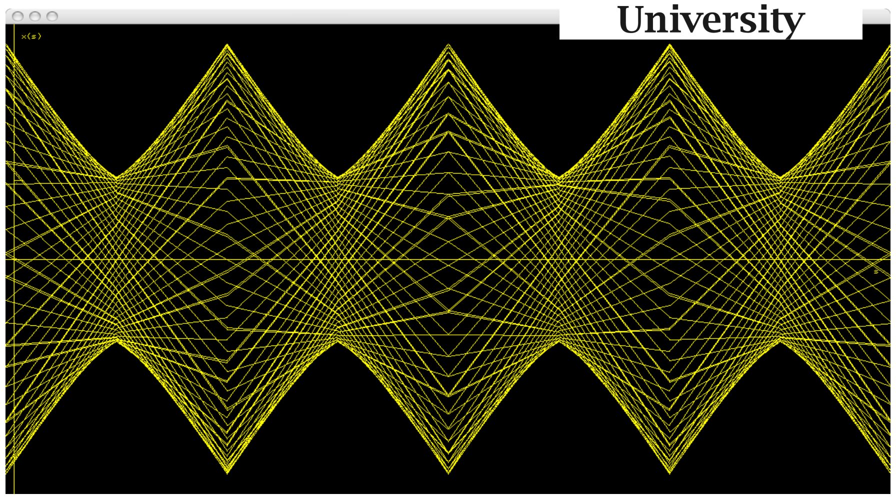

$$x'' + K(s)x = 0$$ Hill's Equation

trial solution: $x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$

requires:

$$\psi' = 1/\beta \quad \longrightarrow \quad \psi(s) = \int \frac{ds}{\beta(s)}$$

and

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \quad \longrightarrow \quad \beta'' + 4K\beta = const.$$

(for *K'* = 0)

for *K* = 0 : $\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$

for *K* > 0 : $\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$

Courant-Snyder Parameters, & Connection to Matrix Approach

- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Previously have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,

$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}$$

- Collectively, β, α, γ are called the *Courant-Snyder Parameters* (sometimes called "Twiss parameters" or "lattice parameters")

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$ *becomes* $$K\beta = \gamma + \alpha'$$

Courant-Snyder Parameters, & Connection to Matrix Approach

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- Collectively, β, α … …ed the *Courant-Snyder Parameters* (sometimes called …ss parameters" or "lattice parameters")

YES! They ARE the same
α, β, γ seen earlier!

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \qquad \textit{becomes} \qquad K\beta = \gamma + \alpha'$$

Solutions using Courant-Snyder Parameters

- Our previous results become

  - drift space:

$$\beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$

$$\rightarrow \quad \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

  - gradient field:

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}} \sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$$

$$\rightarrow \quad \beta(s) = \frac{\beta_0}{2}[1 + \cos(2\sqrt{K}s)] - \frac{\alpha_0}{\sqrt{K}} \sin(2\sqrt{K}s) + \frac{\gamma_0}{2K}[1 - \cos(2\sqrt{K}s)]$$

The Transport Matrix

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- We can always write: $x(s) = a\sqrt{\beta}\sin\Delta\psi + b\sqrt{\beta}\cos\Delta\psi$

- Solve for *a* and *b* in terms of initial conditions and write in matrix form
  - we get:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

So, can write any of our transport matrices in terms of values of C-S parameters at the two end points, and the phase advance between them.

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

Tracking β, α, γ ...

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- Saw earlier that if given values of the Courant-Snyder parameters at one location in the beam line, and if know the matrix of the linear motion between that location and another location downstream, then can compute the values at the second location *via*:

$$K = M\ K_0\ M^T$$

where $K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$

- Have not explicitly proven that the ellipse coefficients found earlier are the SAME as the parameters above, but they are — and, we will.

Evolution of the Phase Advance

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- Also, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase *and* the Courant-Snyder parameters along a beam line from one point to another

Simple Examples

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- Propagation through a Drift:

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = \tan^{-1}\left(\frac{L}{\beta_1 - L\alpha_1}\right)$$

$$\beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha = \alpha_0 - \gamma_0 L$$

$$\gamma = \gamma_0$$

- Propagation through a Thin Lens:

$$M = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = 0$$

$$\beta = \beta_0$$

$$\alpha = \alpha_0 + \beta_0/F$$

$$\gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2$$

$$K = M\ K_0\ M^T$$

- Given *α, β* at one point, can calculate *α, β* at all downstream points

Another Summary

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- So, with knowledge of the layout of (linear) magnetic (and electrostatic) fields from which matrices describing the horizontal and vertical motion can be derived, and with an initial set of Courant-Snyder parameters describing the beam distribution, can *transport* the Courant-Snyder parameters along the beam line
  - Hence, can design a first-order focusing system without having to track particles. Within such a system the beam size will be determined by the value of the emittance used.
- These same C-S parameters describing the beam ellipse in phase space are found to be the same parameters found in the analytical solution to Hill's Equation if we identify

$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta} \qquad \Delta\psi = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$$

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s)+\delta]$$

The Weak Focusing Synchrotron/Betatron

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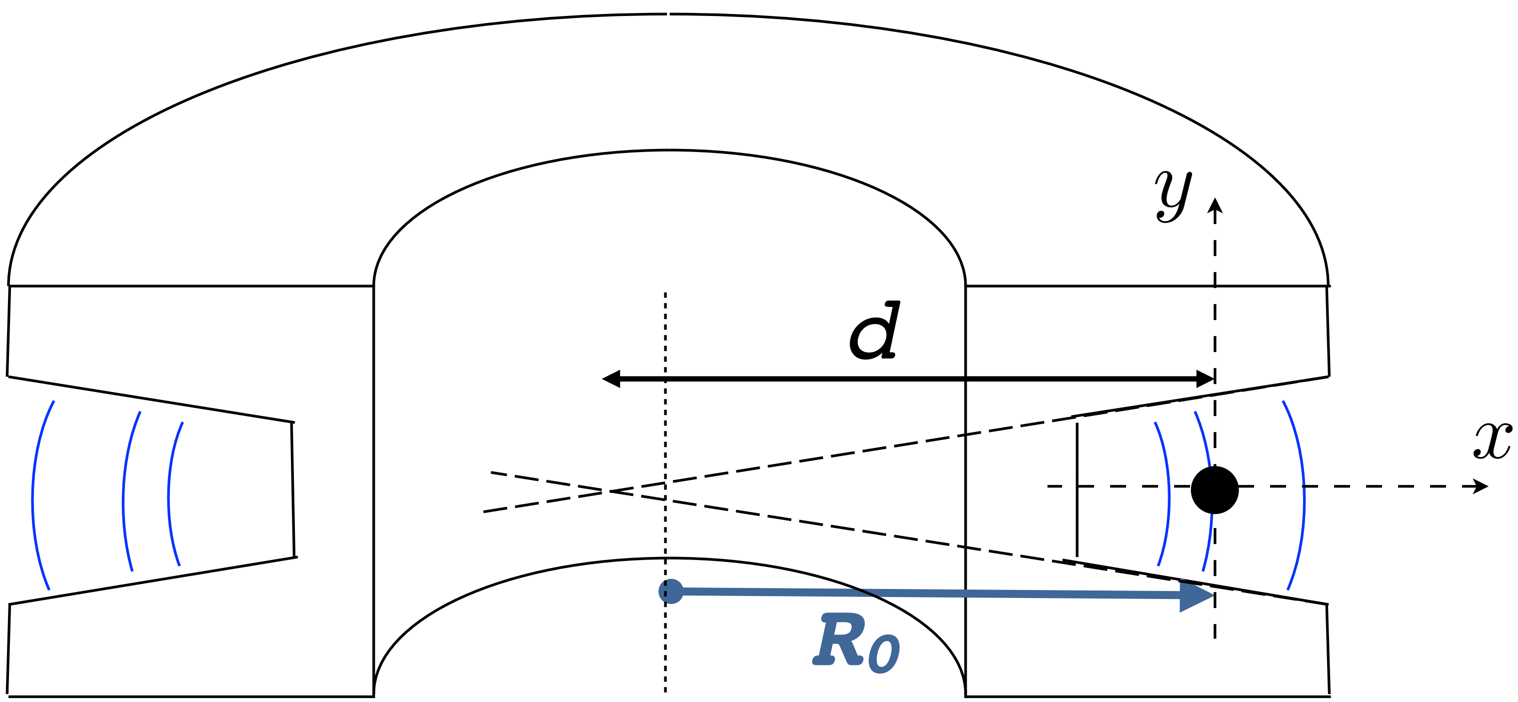
- Early accelerators (betatrons in particular, and early synchrotrons) employed what is now called “weak focusing”



$$B = B_0 \left(\frac{R_0}{r} \right)^n$$

n is determined by adjusting the opening angle between the poles

n = “field index”

$$n \approx \frac{R_0}{d}$$

Let’s look at the stability of transverse motion in this system…

Equation of Motion

- In rotating coordinate system,

$$r = R_0 + x$$

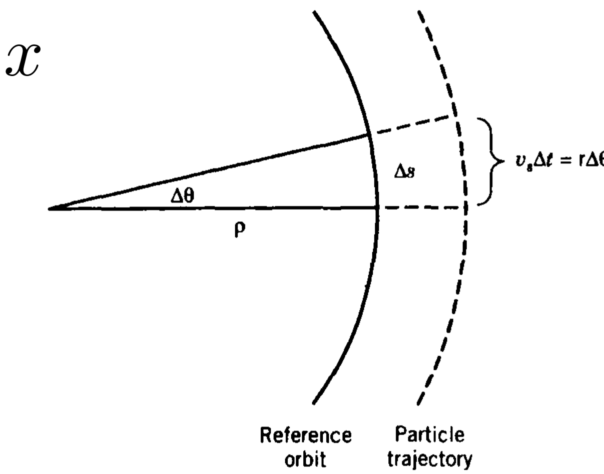


Figure 3.9. Comparison of reference orbit path length ds and particle path length $v_s\,dt$.

$$\frac{d^2x}{ds^2} - \frac{R_0 + x}{R_0^2} = -\frac{eB_y}{p}\left(1 + \frac{x}{R_0}\right)^2$$

Since $v_x \ll v_s$ and $v_y \ll v_s$, to a very good approximation the total momentum p of the particle is $\gamma m v_s$. So

$$\ddot{r} - r\dot{\theta}^2 = -\frac{ev_s^2 B_y}{p}. \qquad (3.38)$$

Now, change to s as the independent variable. Then

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds}, \qquad (3.39)$$

and from Figure 3.9 we see that

$$ds = \rho\, d\theta = v_s\, dt\frac{\rho}{r}. \qquad (3.40)$$

Hence, assuming $d^2s/dt^2 = 0$,

$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2\frac{d^2}{ds^2} = \left(v_s\frac{\rho}{r}\right)^2\frac{d^2}{ds^2}. \qquad (3.41)$$

Replacing r with $\rho + x$, the equation of motion becomes

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2, \qquad (3.42)$$

- Hence,

$$B = B_y(y=0) = B_0\left(\frac{R_0}{r}\right)^n$$

$$B = B_0\left(\frac{1}{1 + x/R_0}\right)^n \approx B_0\left(1 - \frac{n}{R_0}x\right)$$

Stability within a Weak Focusing System

- Thus: $$B_y = B_0 - \frac{nB_0}{R_0}x \qquad B_x = -\frac{nB_0}{R_0}y \qquad (\nabla \times \vec{B} = 0)$$
- So we get, $$x'' + K_x x = x'' + \left(-\frac{nB_0/R_0}{B_0 R_0} + \frac{1}{R_0^2}\right) x = 0$$
- or,

$$x'' + \frac{1-n}{R_0^2}x = 0$$

$$y'' + \frac{n}{R_0^2}\,y = 0$$

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Aperture of Weak Focusing System

- The solutions of the equations of motion are:

$$x'' + \frac{1-n}{R_0^2} x = 0 \quad \longrightarrow \quad x = x_0 \cos(\frac{\sqrt{1-n}}{R_0} s) + x_0' \frac{R_0}{\sqrt{1-n}} \sin(\frac{\sqrt{1-n}}{R_0} s)$$

$$y'' + \frac{n}{R_0^2} y = 0 \quad\quad y = y_0 \cos(\frac{\sqrt{n}}{R_0} s) + y_0' \frac{R_0}{\sqrt{n}} \sin(\frac{\sqrt{n}}{R_0} s)$$

SO, maxima in *x, y* grow with the RADIUS of the accelerator, for a given set of initial beam conditions

Higher energies required larger radii (for ~ constant *B),* and hence the *apertures* had to grow as well

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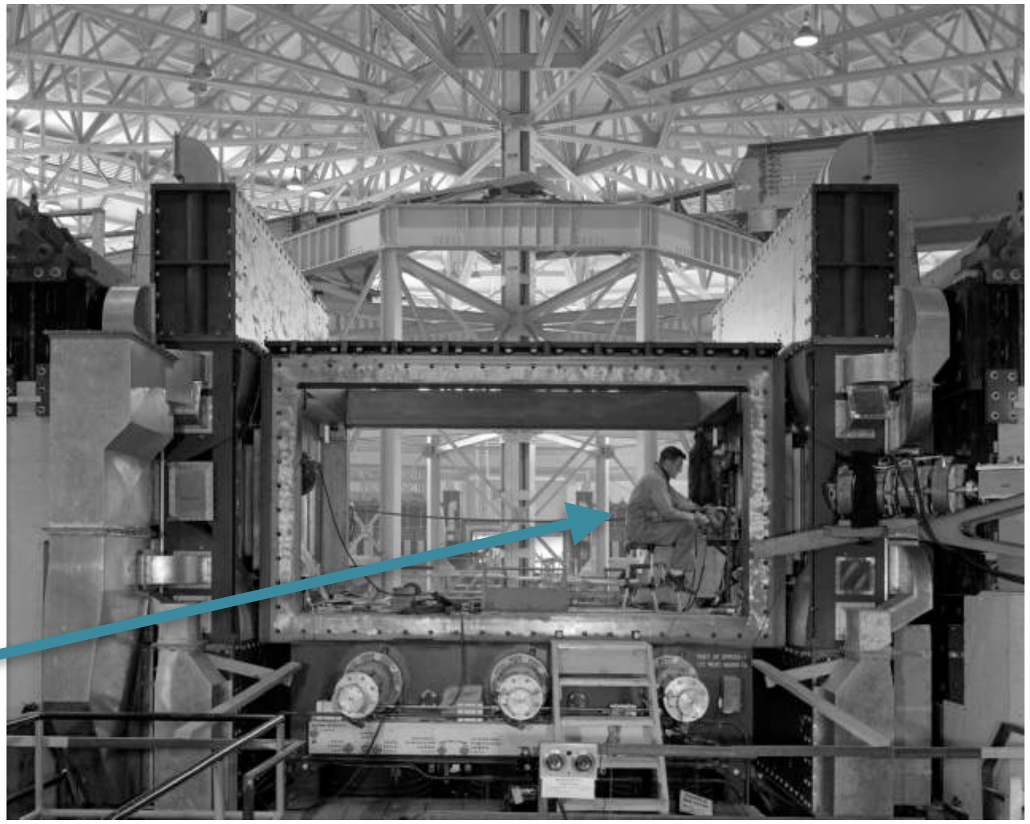
sitting inside the ***beam chamber*** of the Bevatron (LBNL)



Betatron Oscillation Amplitude

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- Transverse oscillations in a synchrotron (or beam line) are called Betatron Oscillations (first observed/analyzed in a "betatron" accelerator)
- Write *x*,*x'* in terms of initial conditions x_0, x'_0 :

$$x(s) = a\sqrt{\beta}\cos\Delta\psi + b\sqrt{\beta}\sin\Delta\psi$$

$$x' = \frac{1}{\sqrt{\beta}}([b - a\alpha]\cos\Delta\psi - [a + b\alpha]\sin\Delta\psi)$$

$$\downarrow$$

$$a = \frac{x_0}{\sqrt{\beta_0}}, \qquad b = \frac{\alpha_0 x_0 + \beta_0 x'_0}{\sqrt{\beta_0}}$$

$$\Longrightarrow x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\,[x_0\cos\Delta\psi + (\alpha_0 x_0 + \beta_0 x'_0)\sin\Delta\psi]$$

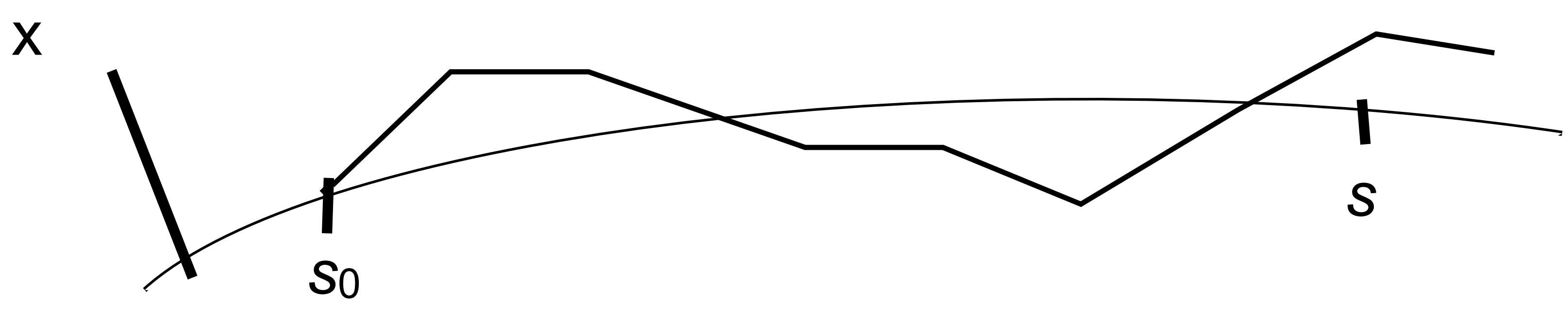
$$\text{amplitude:}\;\; A = \sqrt{\frac{x_0^2 + (\alpha_0 x_0 + \beta_0 x'_0)^2}{\beta_0}}$$

Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle
- Then, downstream, we have

$$\Delta x' = x_0' = \Delta\theta$$

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\,\sin[\psi(s) - \psi_0]$$



Example:
Suppose $\Delta\theta = 0.4$ mrad, $\beta_0 = 4.0$ m, $\beta(s) = 6.4$ m, and $\Delta\psi = n \times 2\pi + 30°$. Then $x(s) = 1$ mm.

Courant-Snyder Invariant

- In general,

$$
\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
&= A\sqrt{\beta}\cos\psi - \alpha x \\
\beta x' + \alpha x &= A\sqrt{\beta}\cos\psi
\end{aligned}
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&= A\sqrt{\beta}\cos\psi - \alpha x \\
\boxed{\beta x' + \alpha x = A\sqrt{\beta}\cos\psi}
\end{aligned}
$$

$$
\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}
$$

Courant-Snyder Invariant

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- In general,

$$
\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
&= A\sqrt{\beta}\cos\psi - \alpha x \\
\boxed{\beta x' + \alpha x} &\boxed{= A\sqrt{\beta}\cos\psi}
\end{aligned}
$$

$$
\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}
$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

Courant-Snyder Invariant

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- In general,

$$
\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
&= A\sqrt{\beta}\cos\psi - \alpha x \\
\beta x' + \alpha x &= A\sqrt{\beta}\cos\psi
\end{aligned}
$$

$$
\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}
$$

$$A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

While C-S parameters evolve along the beam line, the combination above remains constant.

Properties of the Phase Space Ellipse

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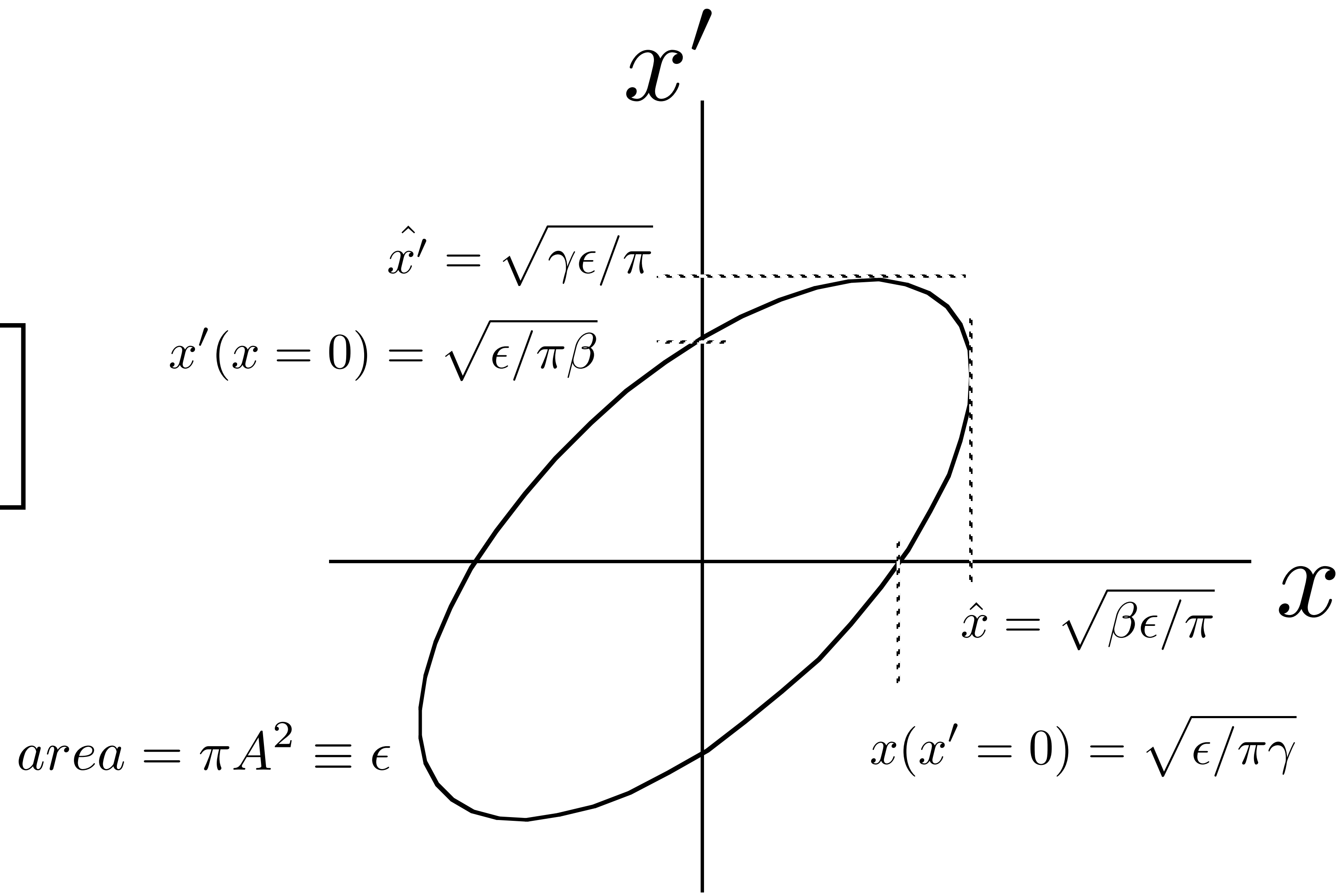
- The initial conditions of a freely-oscillating particle in the beam optics system determine its C-S invariant and hence the particle's phase space ellipse

$$area = \pi A^2$$

while the ellipse changes shape along the beam line, its area remains constant



Emittance = area within a phase space trajectory

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2$$