The Problem

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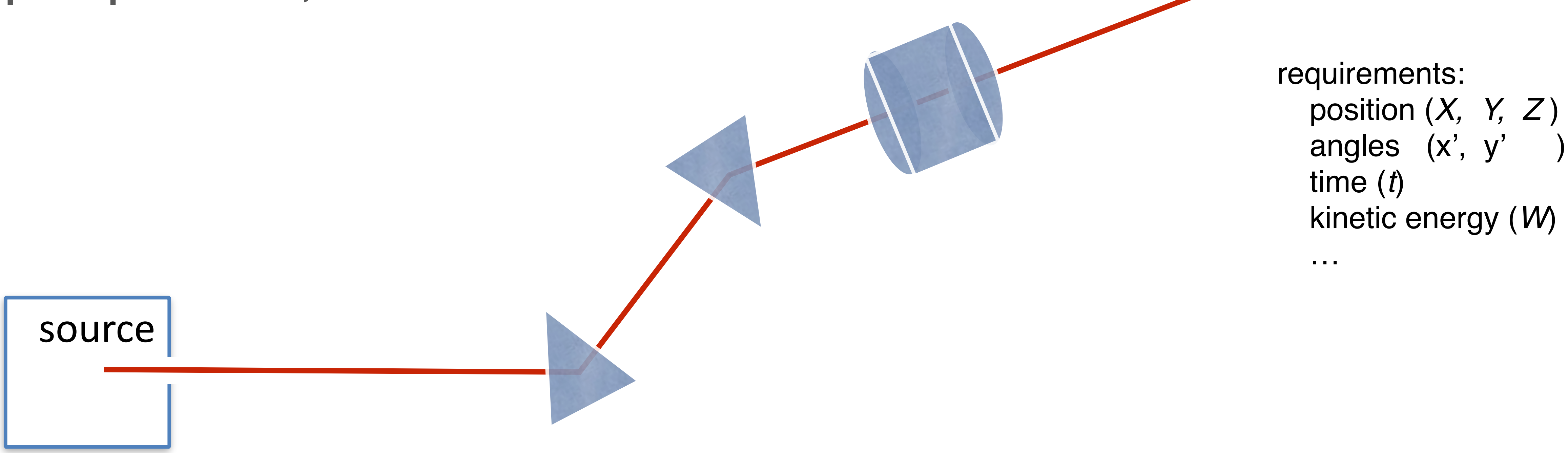
- For given type of particle, create an ideal system to provide particles to a final location with desired trajectory, desired kinetic energy per particle, at the desired time



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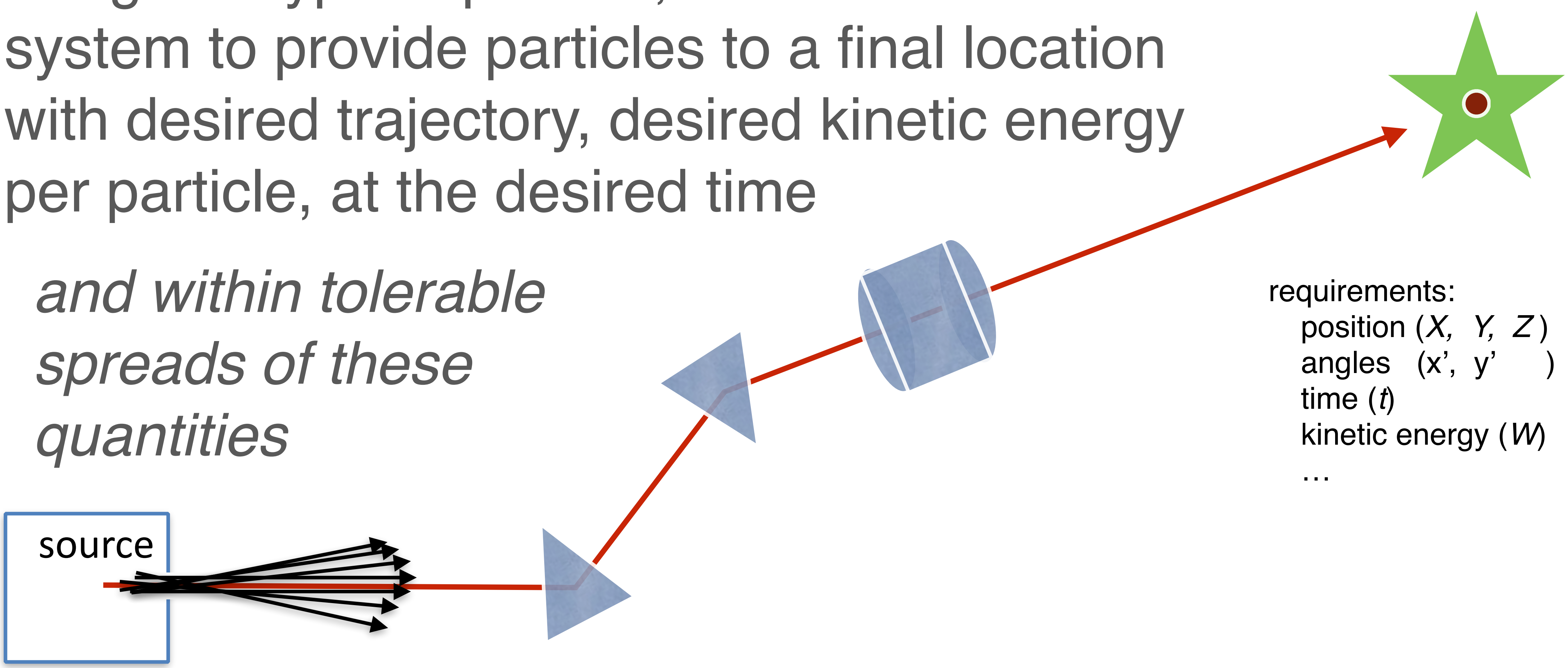

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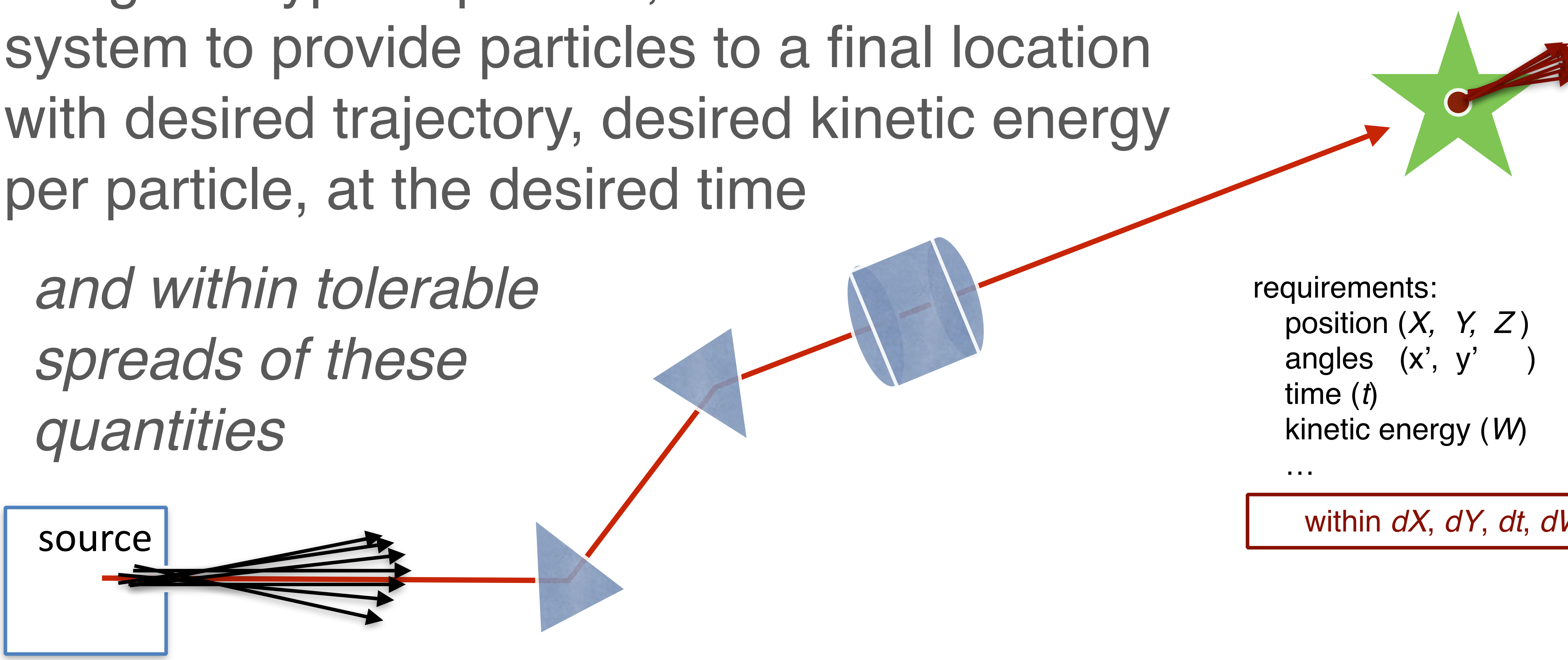

requirements:
position (X, Y, Z)
angles (x', y')
time (t)
kinetic energy (W)
…

within dX, dY, dt, dW, …

Single-Pass vs. Repetitive Systems

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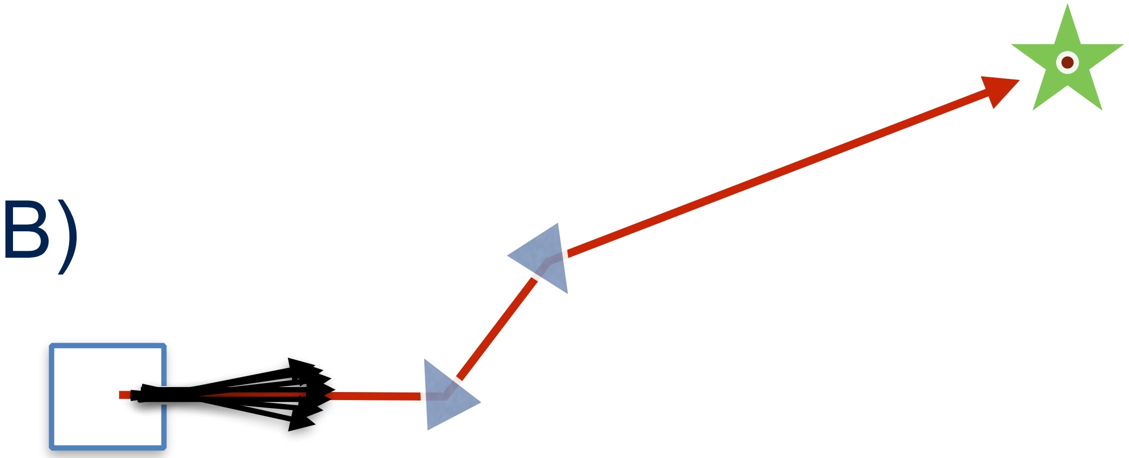
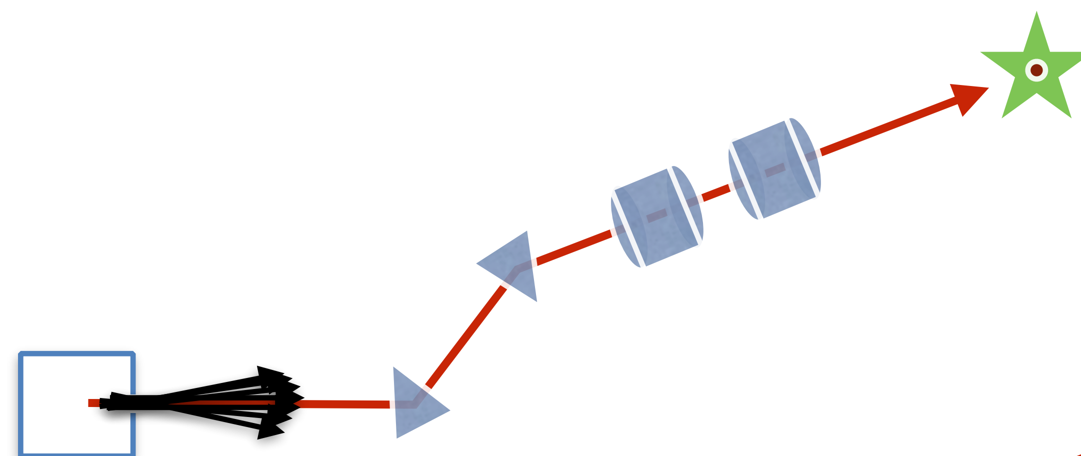
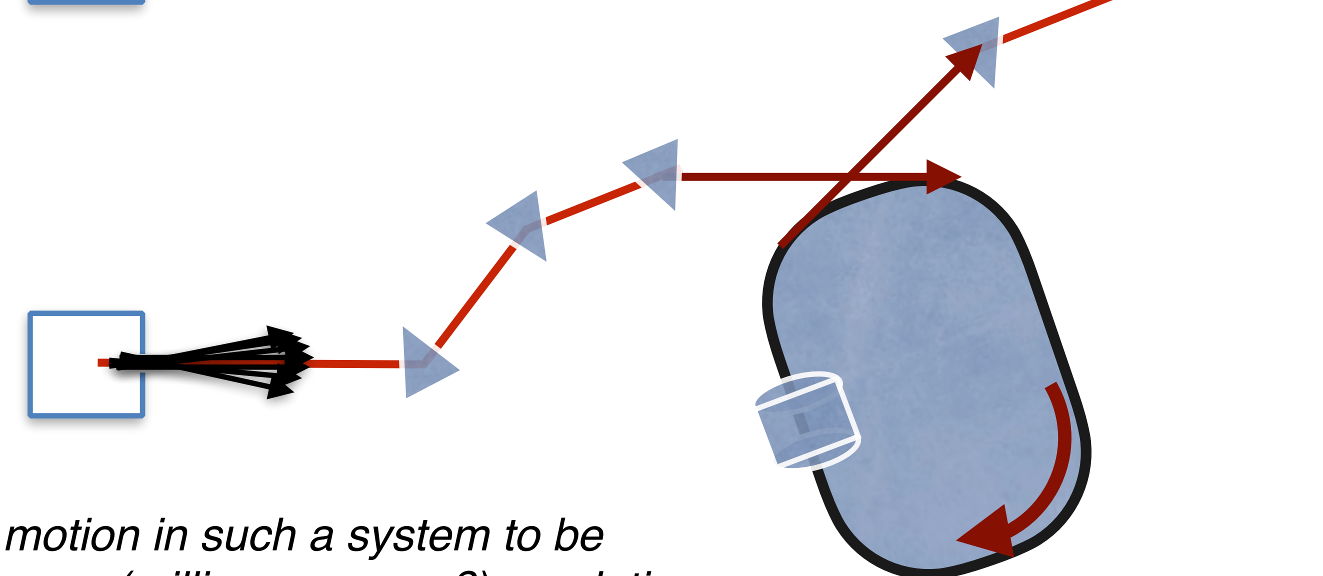
- Beam Transport (from point A to point B)



- Acceleration along the way
  - single-pass with acceleration



  - multi-pass acceleration



may need motion in such a system to be stable for many (millions or more?) revolutions

Stability of Motion Near the Ideal

- Not all particles (any??) begin "on" the design trajectory with *exactly* the ideal energy/momentum
- We wish to have a system that will keep particles near the ideal conditions as they are transported (and possibly accelerated) through the system
- Particles emerge from their "source" with a slight divergence and will need to be guided back toward the ideal trajectory
- Also, particles with different energies/momenta will travel at different speeds, and hence may not arrive at cavities, experiments, etc., at the ideal time

Reduction of the Problem

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- Will treat transverse motion of particles through the accelerator as independent of the longitudinal motion, and study these two cases separately. Must show along the way that this is viable approach.
- Certainly not always be the case...
  - electric fields used for focusing at low energies can also accelerate the particles as well;
  - fields in the gaps of cavities will have focusing effects; etc.
- However, much of the "cross talk" can be minimized, and for much of the particle's journey, especially at higher energies, the major transverse focusing can be performed by magnetic fields -- particle's energy not changed
- Look at "linear" fields, *i.e.* linear restoring forces

Equations of Motion

- Lorentz Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

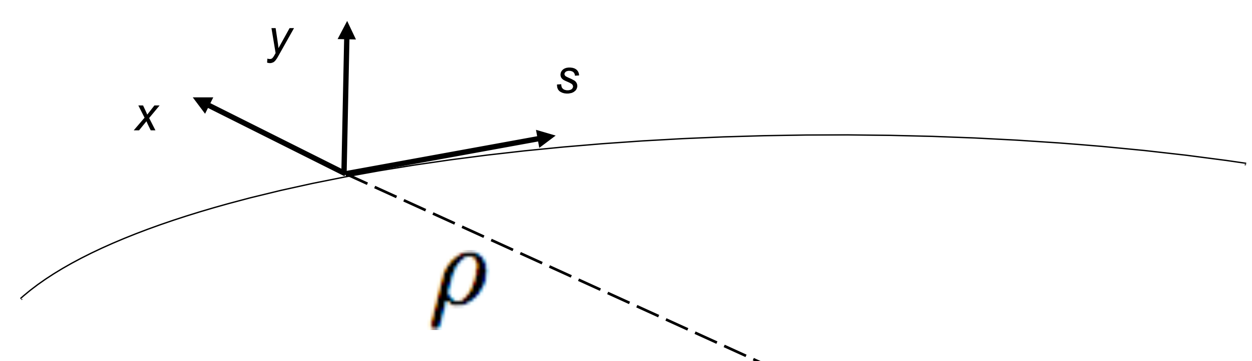
- Magnetic Rigidity
  - particle of unit charge, q = e:

$$B\rho \equiv \frac{p}{q} = \frac{p}{e}$$

  - ion w/ mass A (atomic units, u), charge Q:

$$B\rho = \frac{A}{Q} \left(\frac{1}{300} \frac{\mathrm{T \cdot m}}{\mathrm{MeV/c/u}} \right) p_u$$

- Reference Trajectory
  - Local Coordinate System



Linear Magnetic Fields for Guiding & Focusing

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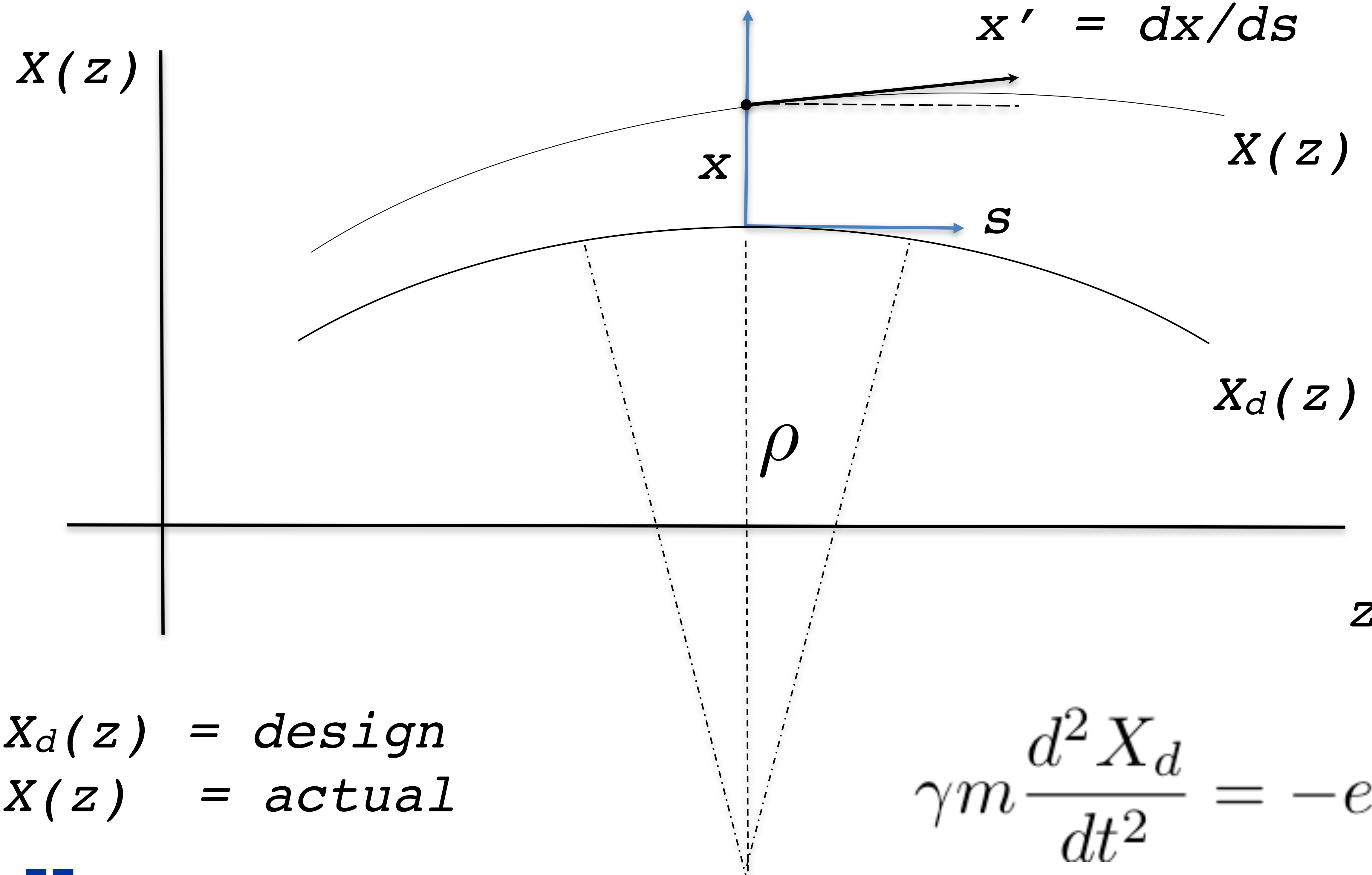

$X_d(z)$ = design
$X(z)$ = actual

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Transverse Fields ($B_z = 0$)

- Drift Space: $B_x = 0$ $B_y = 0$
-
- Bending Region: $B_x = 0$ $B_y = B_0$
  - *(dipole magnet)*
- Focusing Region:
  - *(quadrupole magnet)* $B_x = B'y$ $B_y = B'x$
  - *(electrostatic quadrupole)* $E_y = -E'y$ $E_x = E'x$
- Combined Function Region: $B_x = B'y$ $B_y = B_0 + B'x$
  - *(uniform magnet + ES quad):* $B_x = 0,\ E_y = -E'y$ $B_y = B_0,\ E_x = E'x$
- Accelerating Device: $B_x = 0,\ B_y = 0;$ $E_z = V/g$
  - *(cavity)*

Linear Restoring Forces

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- Assume linear guide fields: --

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

- Look at radial motion:

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x' v_s$$

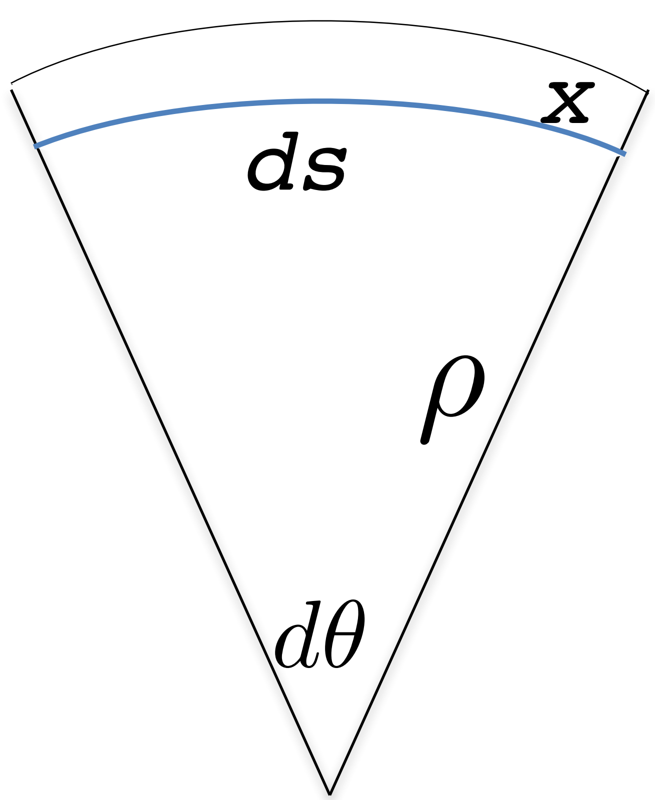
$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{e v_s^2 B_y}{p}$$

$$v_s^2 \left(\frac{\rho}{r}\right)^2 x'' - (\rho + x)\left(\frac{v_s}{r}\right)^2 = -\frac{e v_s^2 B_y}{p}$$

$$dt = \frac{\rho + x}{\rho}\frac{ds}{v_s}$$

$$\frac{ds}{dt} = v_s \frac{\rho}{r}$$



$$x'' - \frac{\rho + x}{\rho^2} = -\frac{e B_y}{p}\left(\frac{r}{\rho}\right)^2$$

linearize…

$$x'' - \frac{1}{\rho} - \frac{x}{\rho^2} = -\frac{B_0 + B'x}{B\rho}\left(1 + \frac{2x}{\rho} + \frac{x^2}{\rho^2}\right)$$

$$x'' + \left(\frac{1}{\rho^2} + \frac{B'}{B\rho}\right) x = 0$$

Hill's Equation

- Now, for vertical motion:

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

$$y'' = \frac{eB_x}{p}\left(\frac{r}{\rho}\right)^2$$

$$y'' - \frac{eB_x}{p}\left(1 + \frac{x}{\rho}\right)^2 = 0$$

- So we have,
  - to lowest order,

linearize…

$$y'' - \frac{eB'y}{p} = 0$$

$$x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right)x = 0$$
$$y'' - \left(\frac{B'}{B\rho}\right)y = 0$$

Hill's Equation

General Form: → $$x'' + K(s)x = 0$$

Piecewise Method of Solution

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- Hill's Equation: $$x'' + K(s)x = 0$$

- Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: drift $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$

- *K*

- *K*

Here, x refers to horizontal or vertical motion, with relevant value of K

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- *K* > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$

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- *K* < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

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Quad, Gradient Magnet, edge, ...

- *K* < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

Here, x refers to horizontal or vertical motion, with relevant value of K

Piecewise Method -- Matrix Formalism

- Write solution to each piece in matrix form
  - for each, assume K = const. from s=0 to s=L

- $K = 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K > 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K < 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Determining *K* — *Examples*

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- Quadrupole Magnets

$$K_x = \frac{B'}{B\rho}$$
$$K_y = -\frac{B'}{B\rho}$$

- Sector Bend Dipole Magnets

$$K_x = \frac{1}{\rho^2}$$
$$K_y = 0$$

$$x'' + K_x\, x = 0$$
$$y'' + K_y\, y = 0$$

- Sector Bends with Electrostatic Focusing

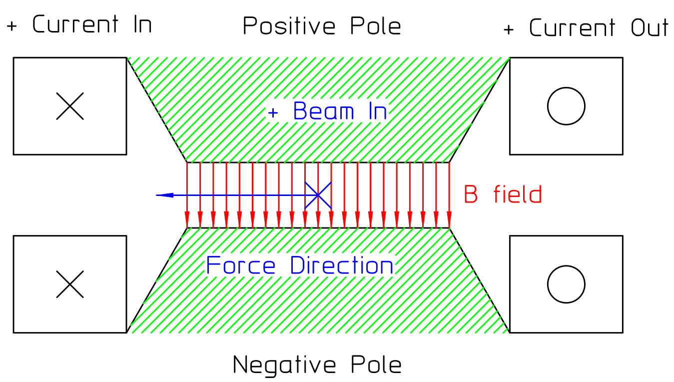
$$K_x = \frac{1}{\rho^2} - \frac{E'}{v(B\rho)}$$
$$K_y = \frac{E'}{v(B\rho)}$$

← *g*-2 arrangement

- Other Considerations
  - Combined function magnet
  - Rectangular Bend
  - Bend with arbitrary Edge Angles

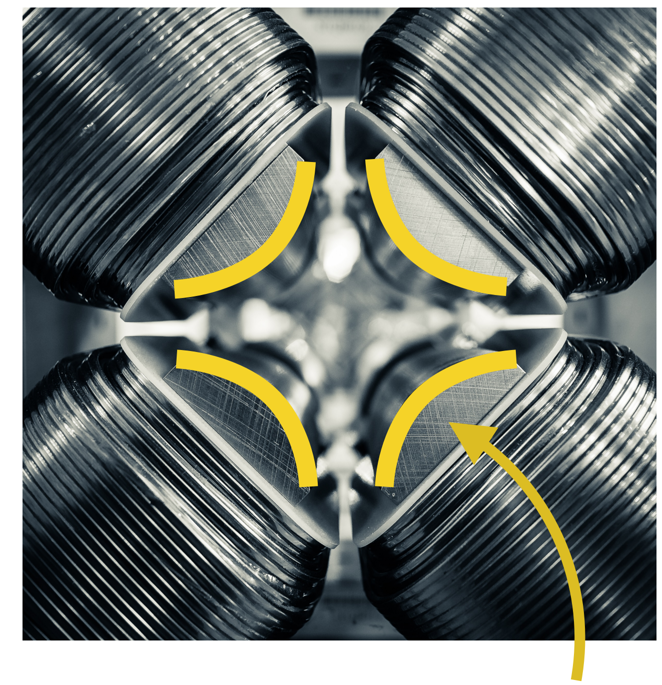
The Quadrupole Magnet

- Dipole magnet:
  - uniform bend field




Dipole, FNAL

- Quadrupole magnet:
  - field = 0 on longitudinal axis
  - varies linearly with transverse position



$\Phi_m = constant$

$$\vec{B} = \nabla\Phi_m \longrightarrow \nabla^2\Phi_m = 0$$

$$\Phi_m = Cr^n \sin n\phi$$

$$B_r = nCr^{n-1} \sin n\phi$$

$$B_\phi = nCr^{n-1} \cos n\phi$$

2n-pole magnet
$n = 2$ (quadrupole):

$$B_x = 2Cy \equiv B'y$$

$$B_y = 2Cx \equiv B'x$$

"Thin Lens" Quadrupole

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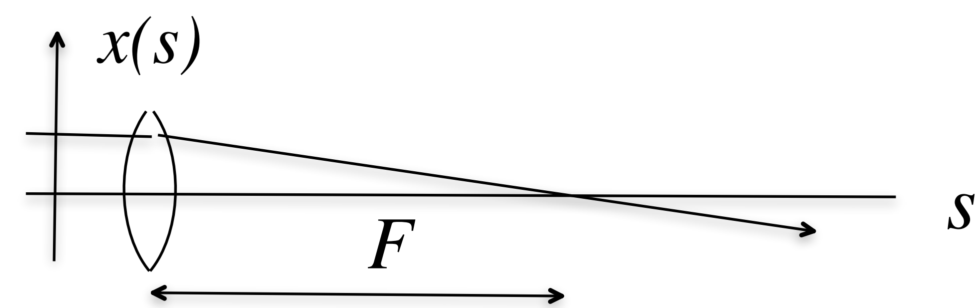
- If quadrupole magnet is short enough, particle's offset through the quad does not change by much, but the slope of the trajectory does -- acts like a "thin lens" in geometrical optics



- Take limit as *L* --> 0, while *KL* remains finite

  - (similarly, for defocusing quadrupole)

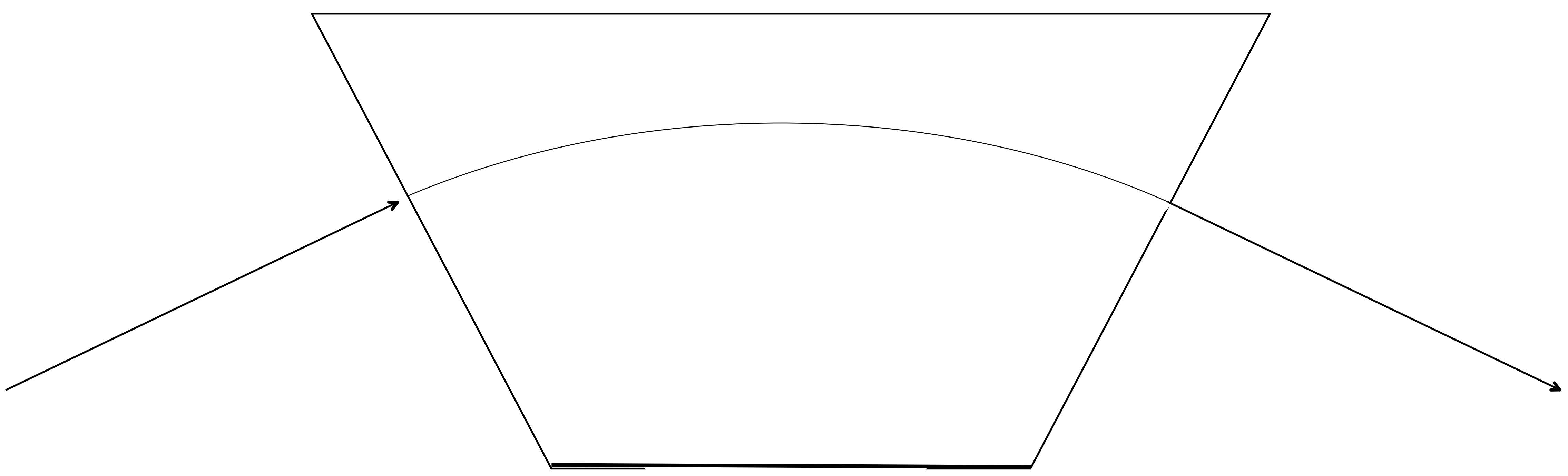
$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

- Valid approx., if *F* >> L

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

Sector Magnets

- Sector Dipole Magnet: "edge" of magnetic field is perpendicular to incoming/outgoing design trajectory:



Field points "*out of the page*"

Sector Magnets & Sector Focusing

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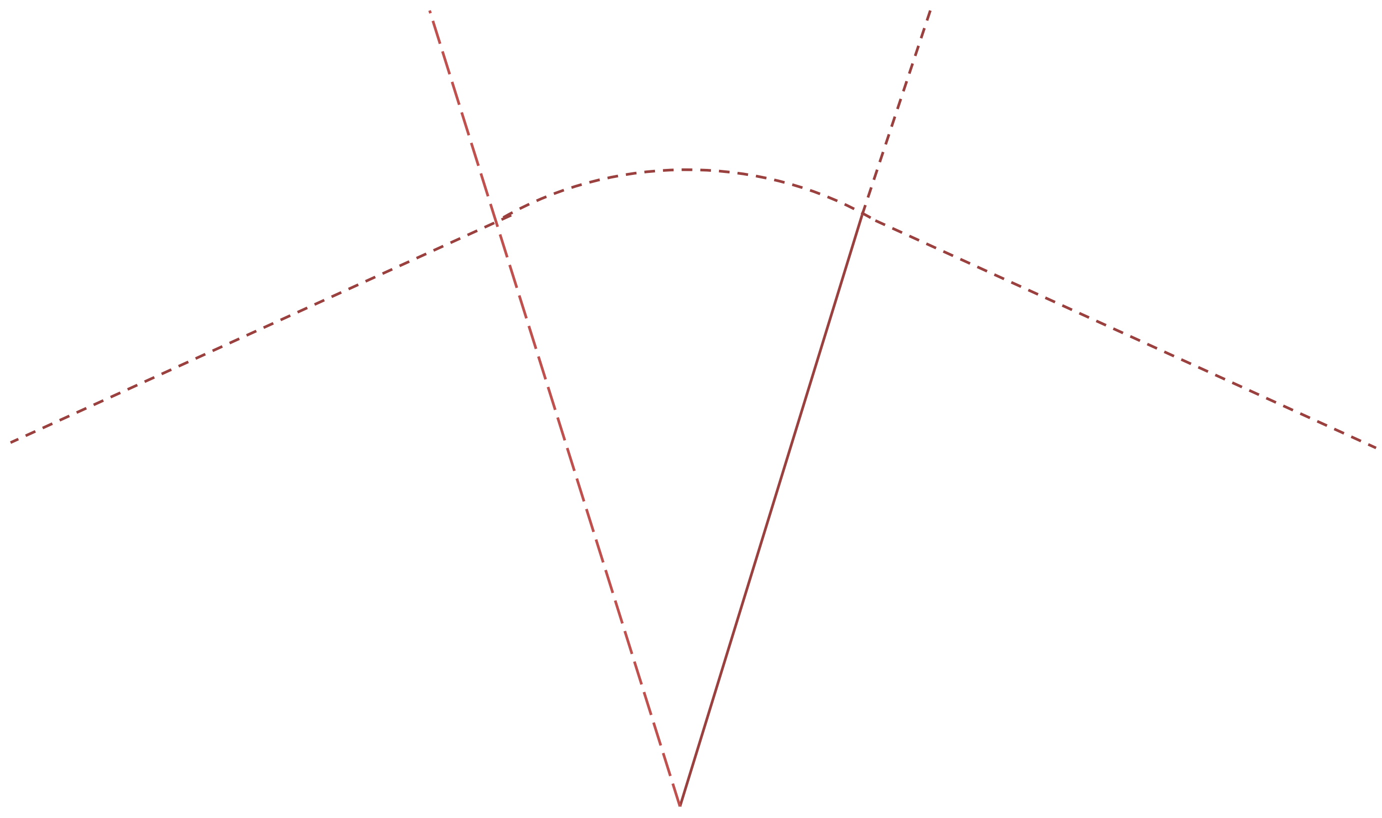
- Incoming ray displaced from ideal trajectory will experience more/less bending field, thus is "focused" toward axis in the bend plane:



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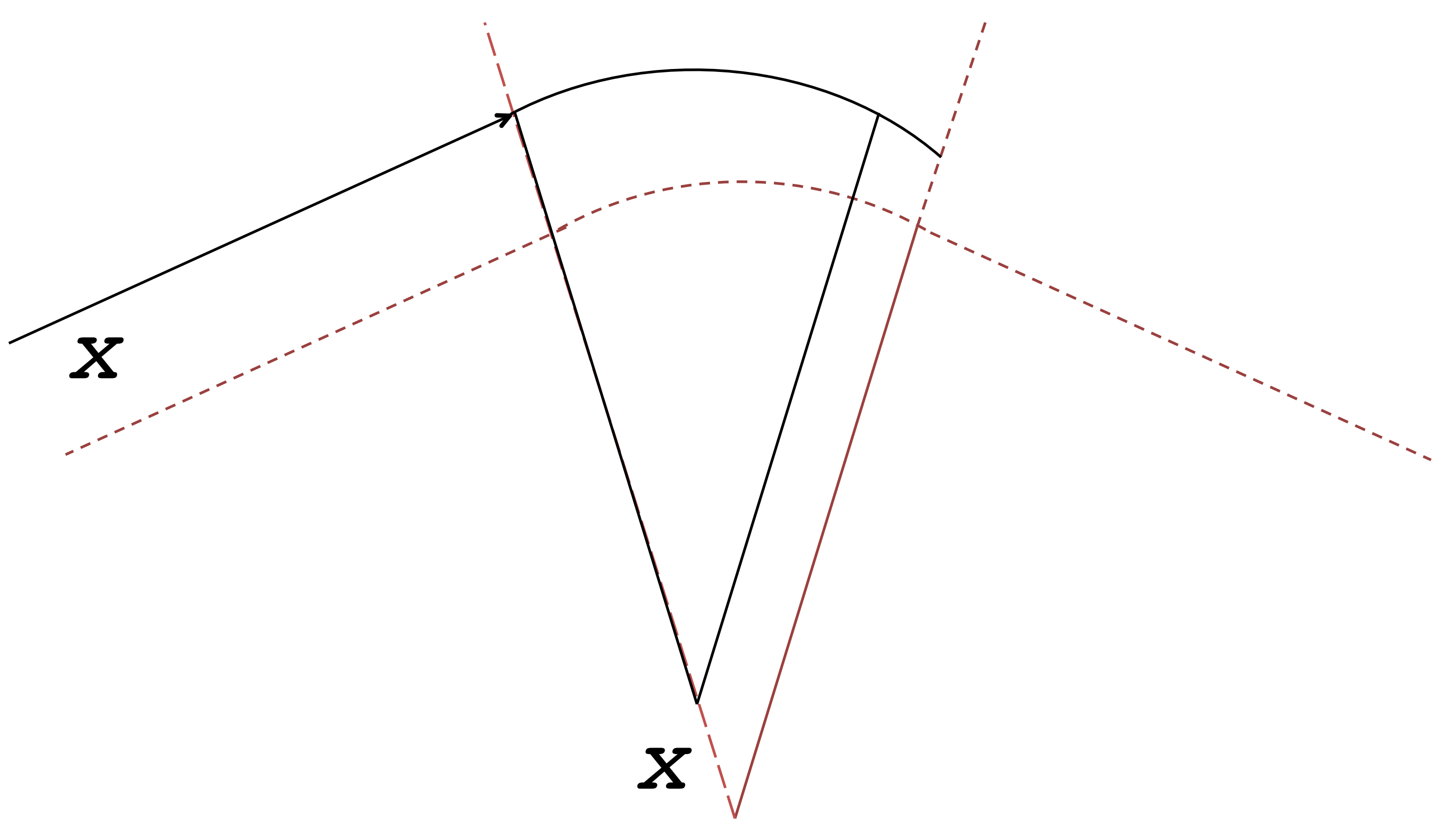

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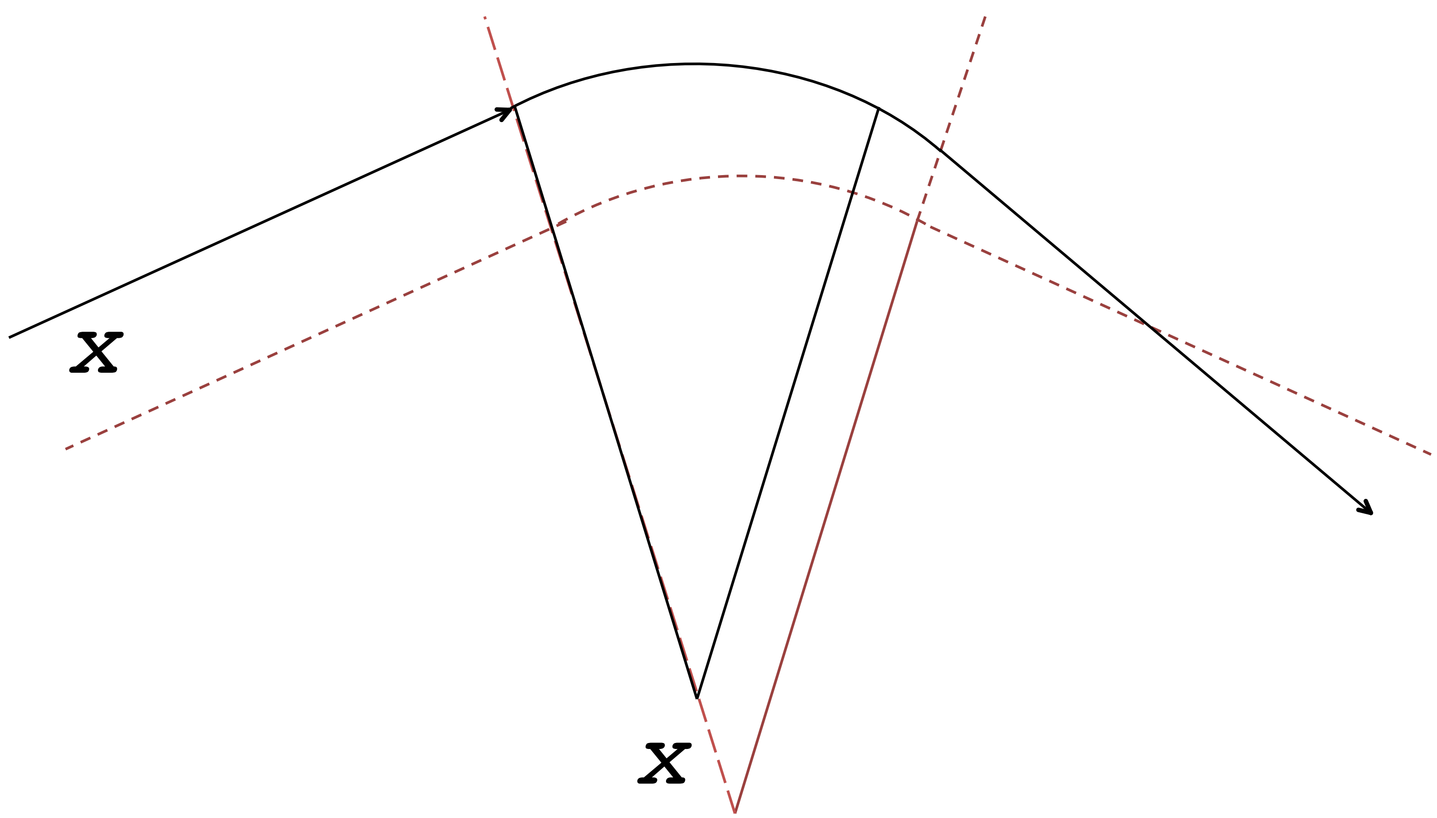

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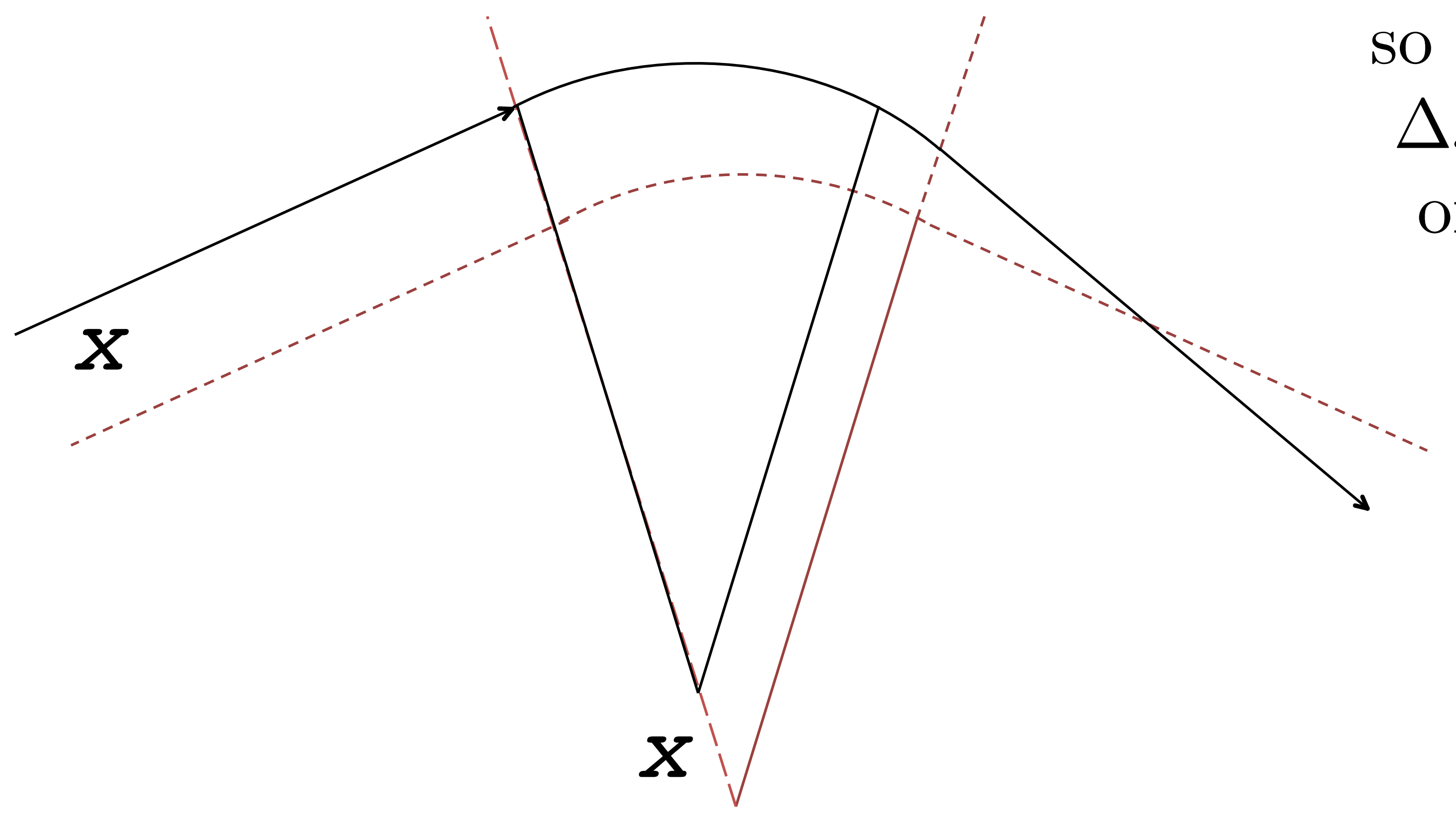

$$\text{Extra path length} = \Delta s = x\ \theta$$
$$\text{so extra bend angle} = \Delta x' = -\Delta s/\rho$$
$$\Delta x' = -(\theta/\rho)x = -(\ell/\rho^2)x$$
$$\text{or, } x'' = dx'/ds = -(1/\rho^2)x$$

$$\text{Thus, } K_x = 1/\rho^2,\ K_y = 0.$$

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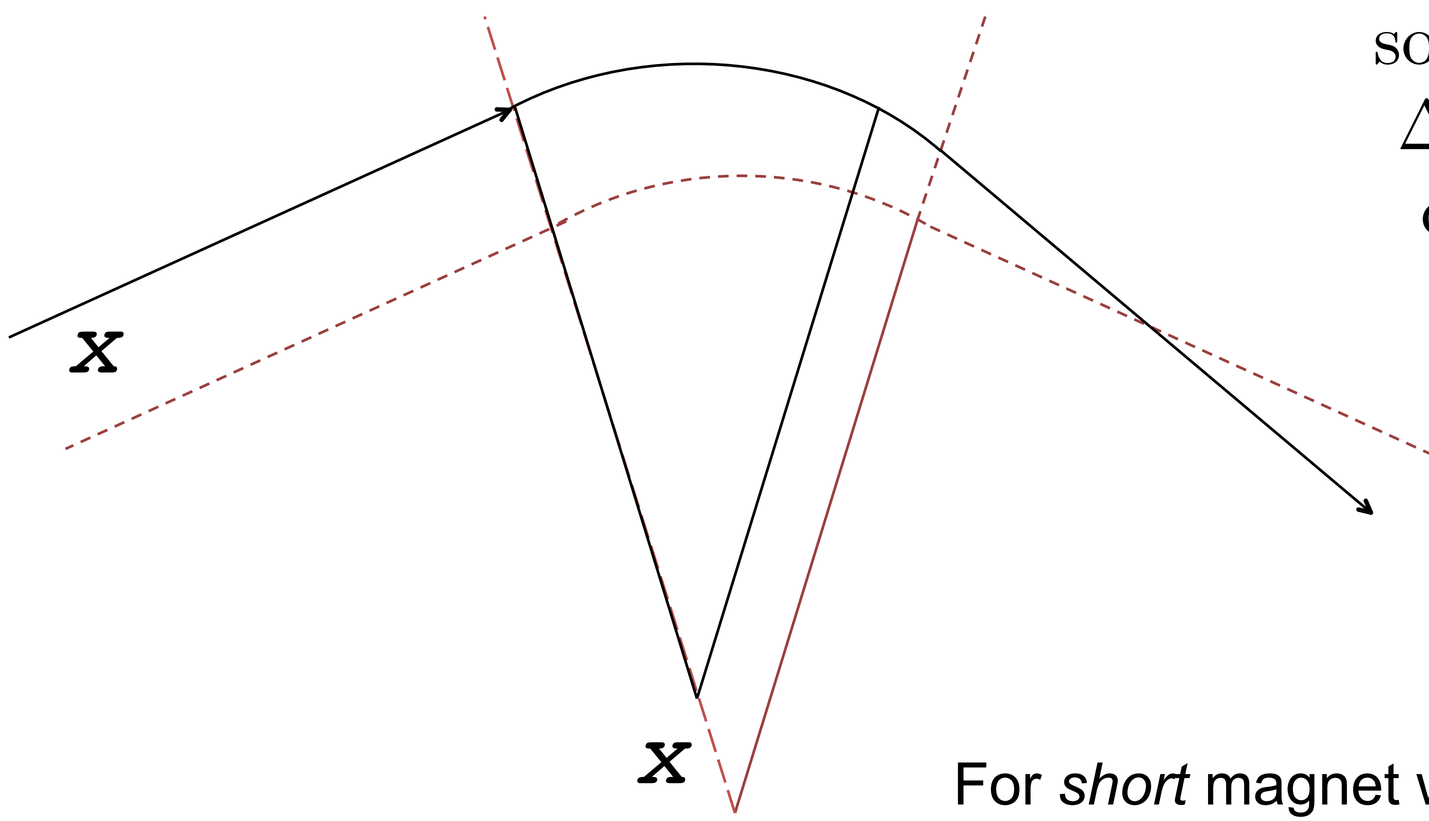

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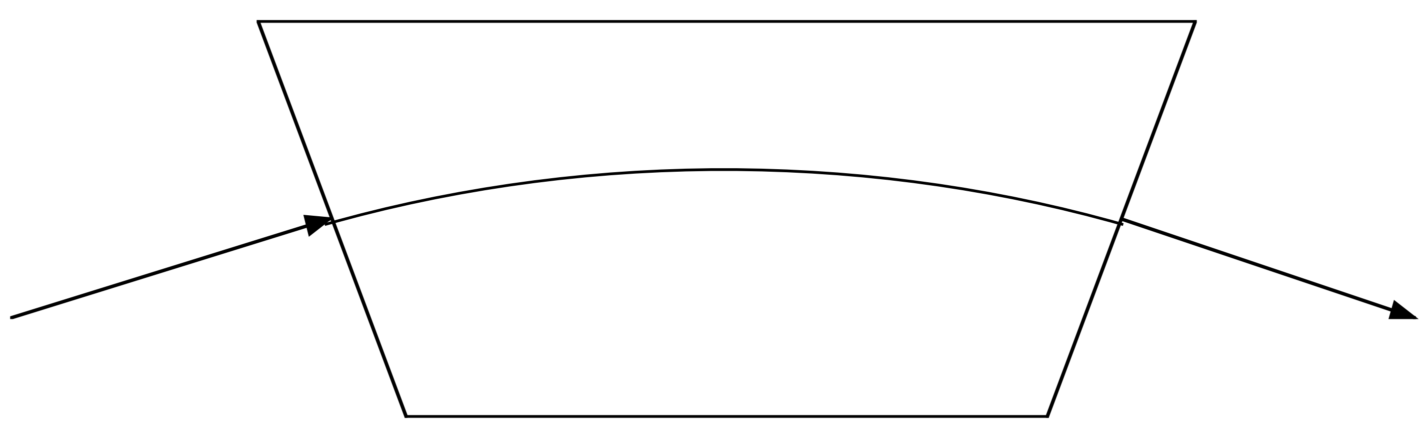
For *short* magnet with small bend angle, will also acts like lens in the bend plane ⇨ $$\frac{1}{f_x} = \frac{\theta}{\rho}$$

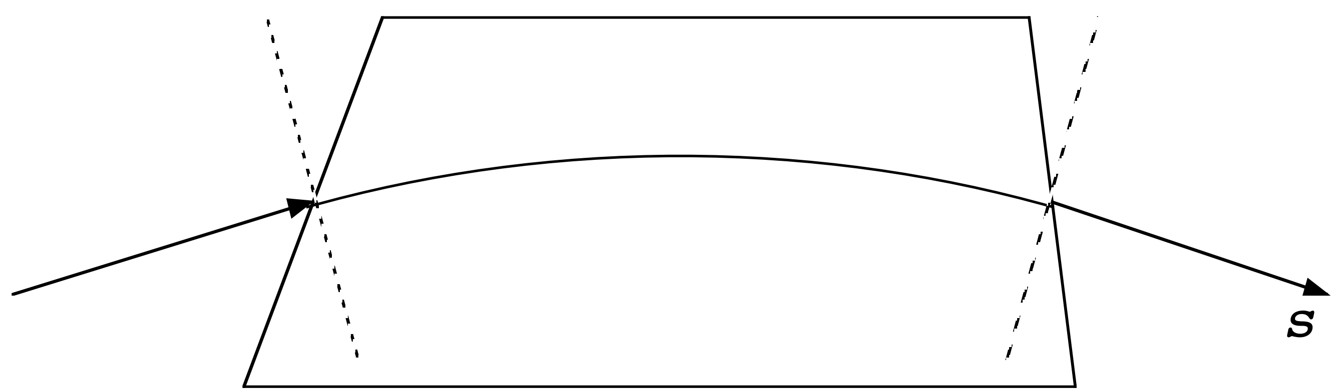
Edge Focusing

- In an ideal *sector magnet*, the magnetic field begins/ends exactly at $s = 0, L$ independent of transverse coordinates x, y relative to the design trajectory.
  - *i.e.*, the face of the magnet is perpendicular to the design trajectory at entrance/exit



Edge Focusing

- However, could (and often do) have the faces at angles w.r.t. the design trajectory -- provides "edge focusing"



- Since our transverse coordinate *x* is everywhere perpendicular to *s*, then a particle entering with an offset will see more/less bending at the interface...

Rectangular Bending Magnet

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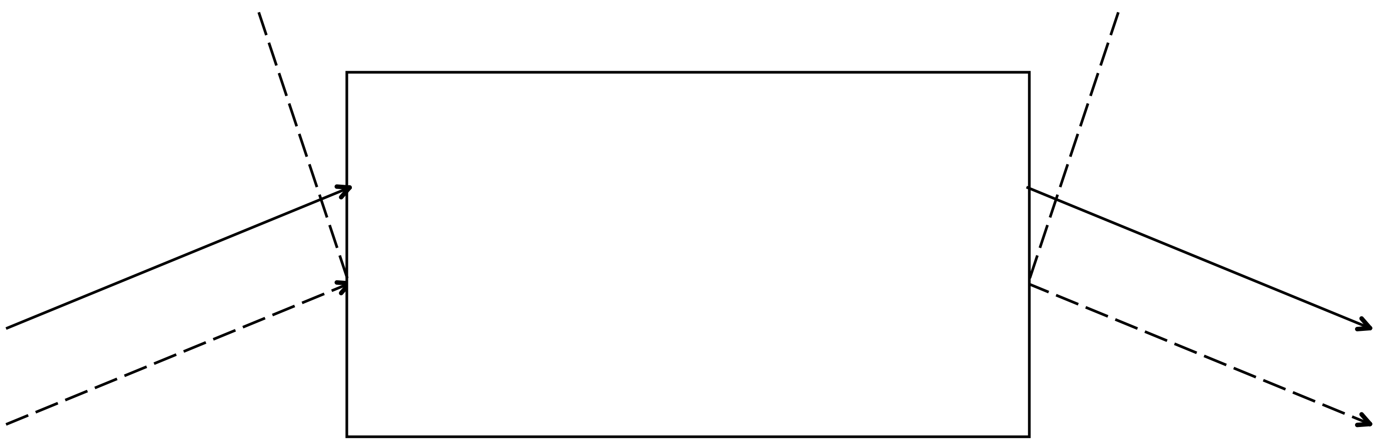


In the bending plane, each edge acts as a defocusing lens with focal length:

$$\frac{1}{f} = -\frac{\theta/2}{\rho} = -\frac{\theta}{2\rho}$$

For Pure Sector Magnet,

hor: $$\frac{1}{f_x} \approx \frac{\theta}{\rho}$$

ver: $$\frac{1}{f_y} \approx 0$$

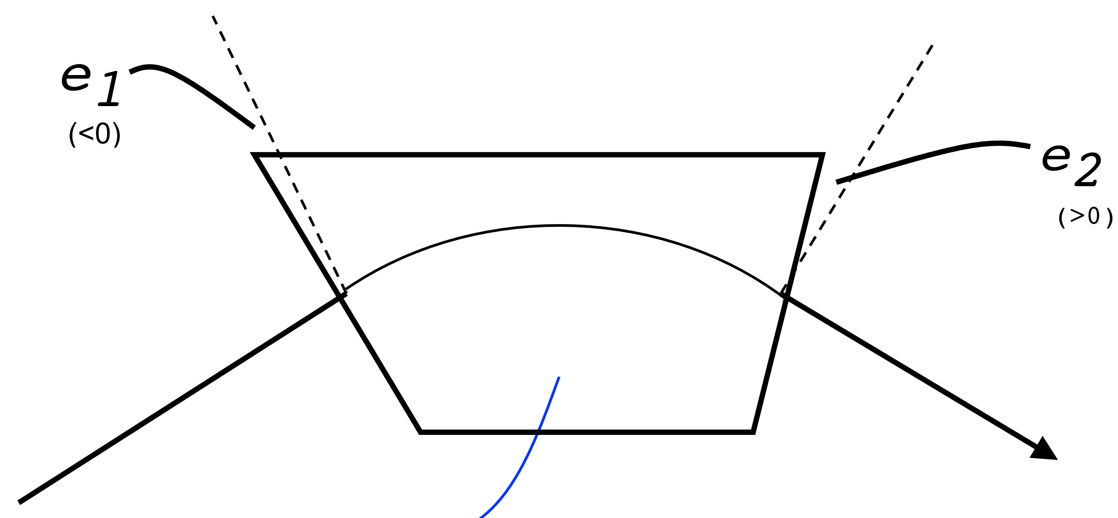
For Rectangular Magnet,

hor: $$\frac{1}{f_x} \approx -\frac{\theta}{2\rho} + \frac{\theta}{\rho} - \frac{\theta}{2\rho} = 0$$

ver: $$\frac{1}{f_y} \approx \frac{\theta}{2\rho} + \frac{\theta}{2\rho} = \frac{\theta}{\rho}$$

Transport through a General Bending Magnet

- Put all the pieces together…



- $M_{total} = M_{e2}\, M_{body}\, M_{e1}$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan e_2}{\rho} & 1 \end{pmatrix} \begin{pmatrix} \cos(\ell/\rho) & \frac{1}{\rho}\sin(\ell/\rho) \\ -\frac{1}{\rho}\sin(\ell/\rho) & \cos(\ell/\rho) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\tan e_1}{\rho} & 1 \end{pmatrix}$$

$$M_y = \begin{pmatrix} 1 & 0 \\ -\frac{\tan e_2}{\rho} & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\tan e_1}{\rho} & 1 \end{pmatrix}$$

Comments

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- Very often, especially for highly-relativistic particles, the bend radii with bending magnets can be large, and hence the sector focusing can be a small effect. However, in accelerators with dozens, hundreds, or thousands of elements, it can certainly add up.
- Same can be said for edge effects in many circumstances.
- One must always seek to understand the particular situation and determine what assumptions can be made for the level of detail one is studying.

Piecewise Method -- Matrix Formalism

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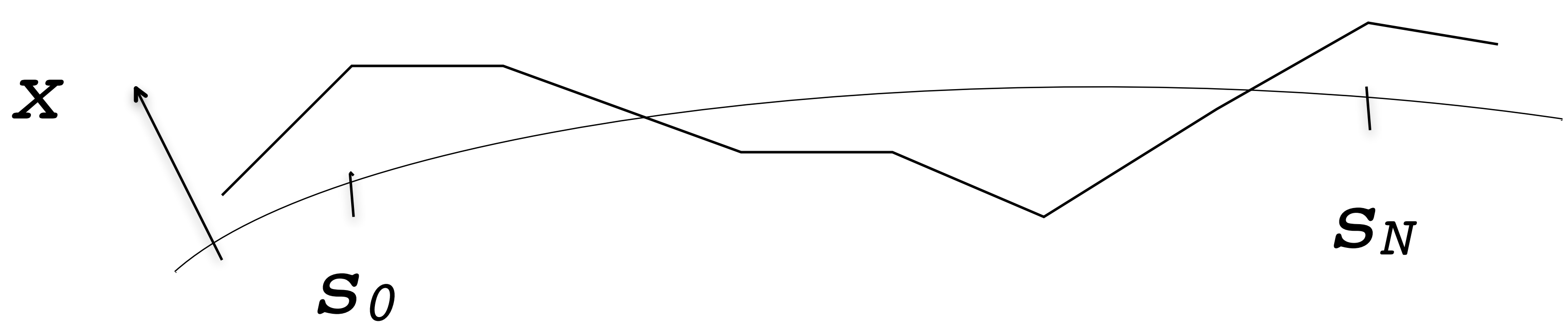
- Arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Example: A Beam Line Calculation

- Will consider two particle trajectories, starting with
  - (x,x') = (0, 0.5 mrad), and (x,x') = (5 mm, 0)
- A distance 6 m later, the trajectories enter a thin lens quadrupole of focal length F = 3 m. This is followed by a second quadrupole of focal length -F, a distance 1 m later.

  - Find the trajectories (x,x') for each case at the exit of the second quad

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$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 1\ m \\ \frac{1}{3\ m} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ -\frac{1}{3\ m} & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 5\ m \\ -\frac{1}{9\ m} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

x_0 = 0 mm, x'_0 = 0.5 mr —> x = 2.5 mm, x' = 0.33 mr

x_0 = 5 mm, x'_0 = 0.0 mr —> x = 3.3 mm, x' = -0.6 mr

Can now make LARGE accelerators!

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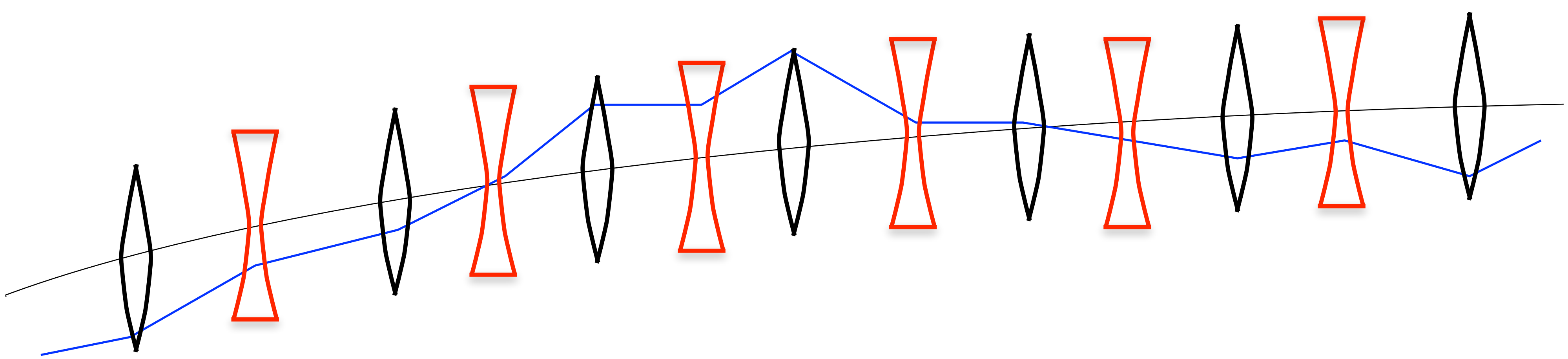
- Since the lens spacing can be made arbitrarily short, with corresponding focusing field strengths, then in principal can make beam transport systems (and linacs and synchrotrons, for instance) of arbitrary size

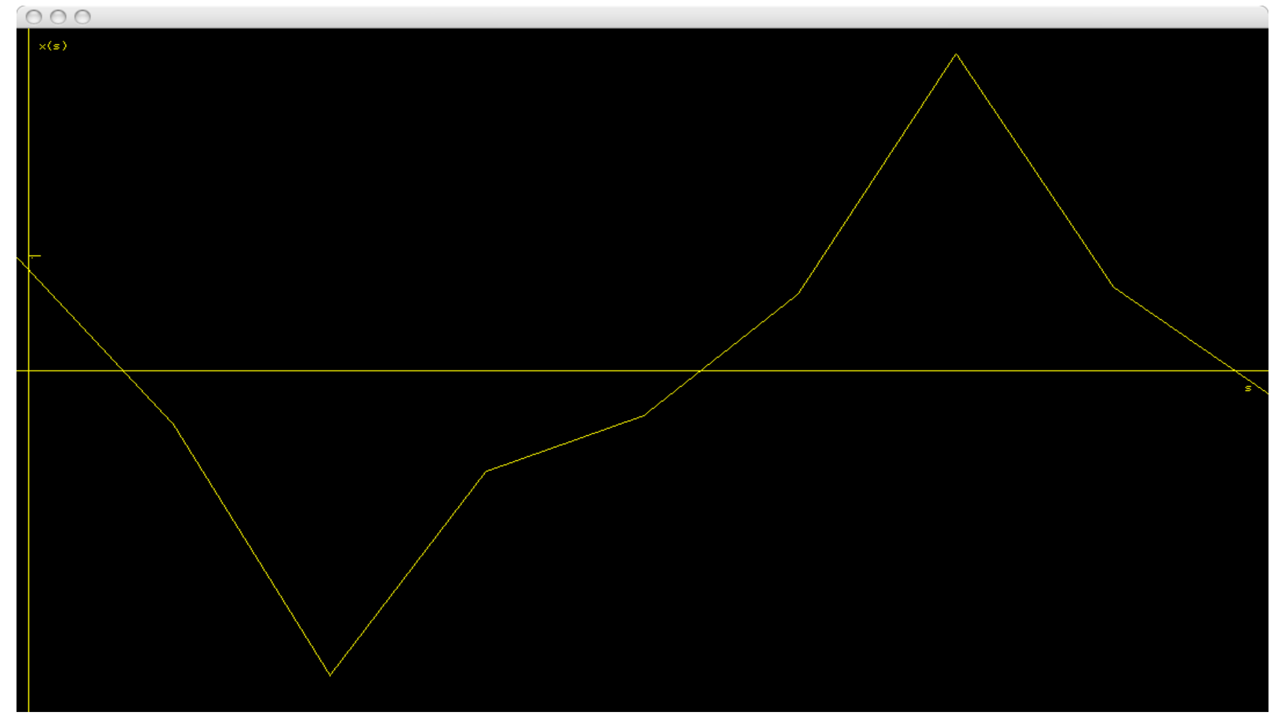


- Instrumental in paving the way for very large accelerators, both linacs and especially synchrotrons, where the bending and focusing functions can be separated into distinct magnet types

The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line



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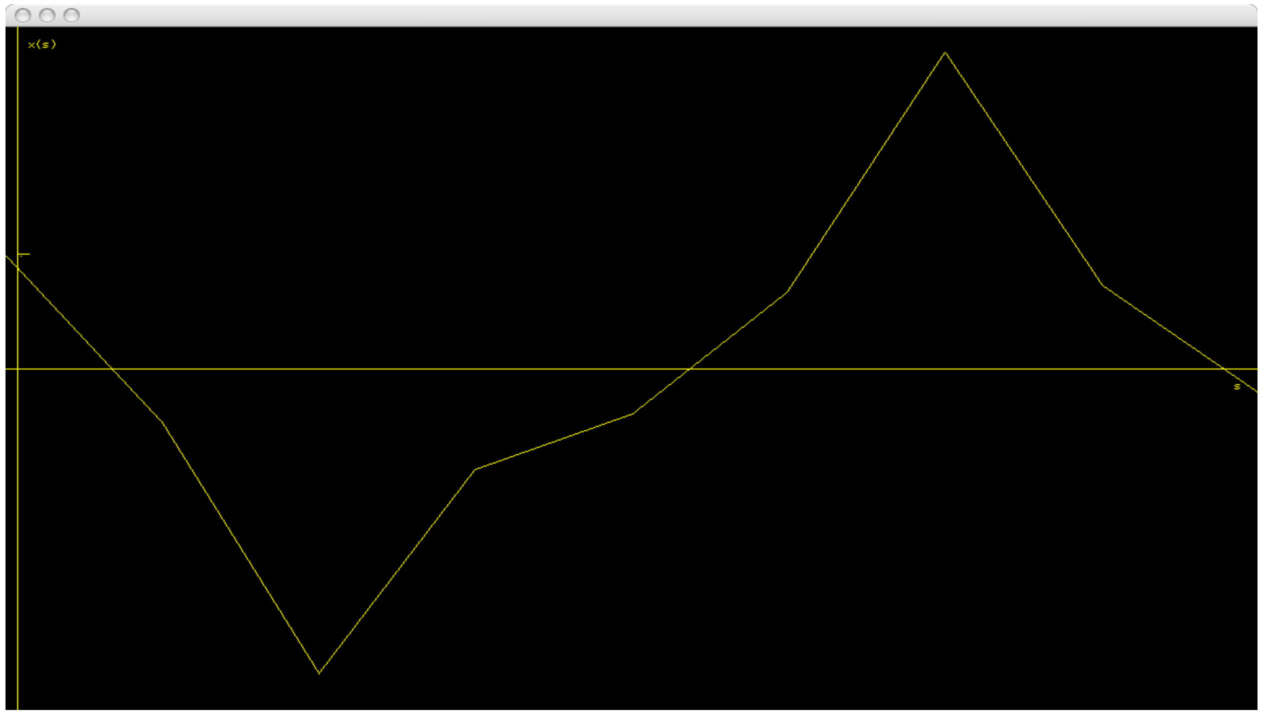
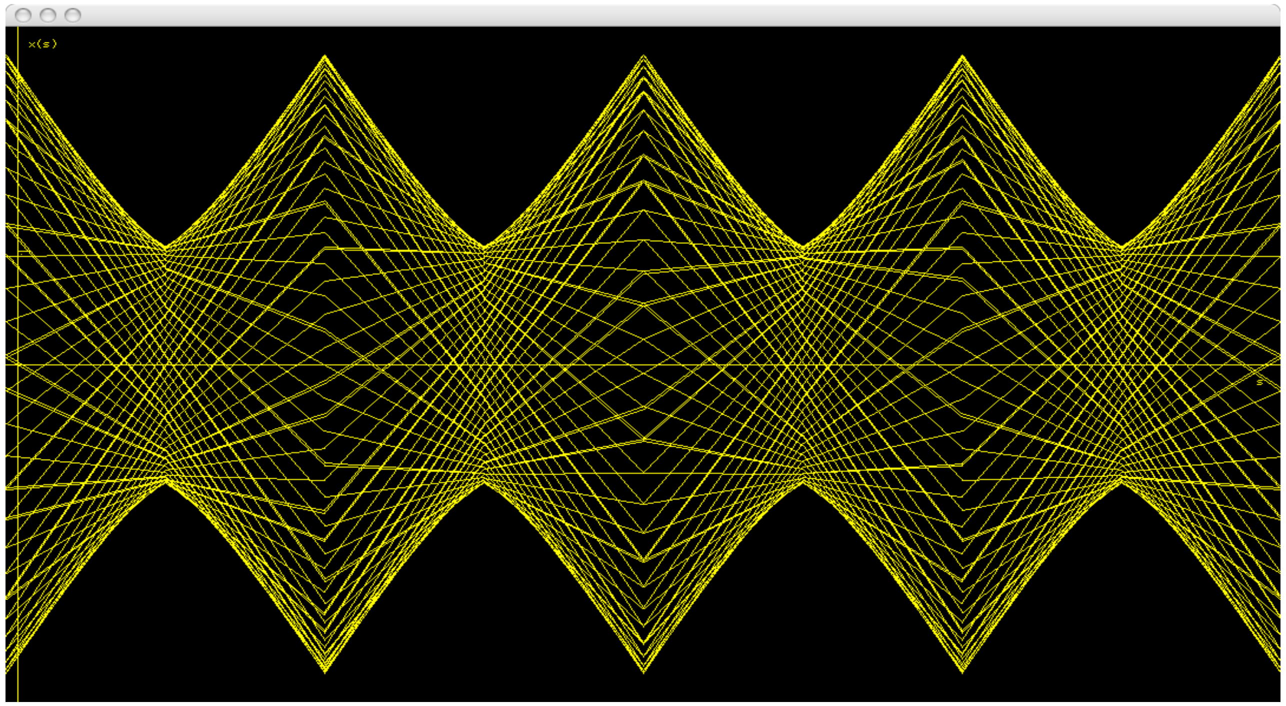
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line

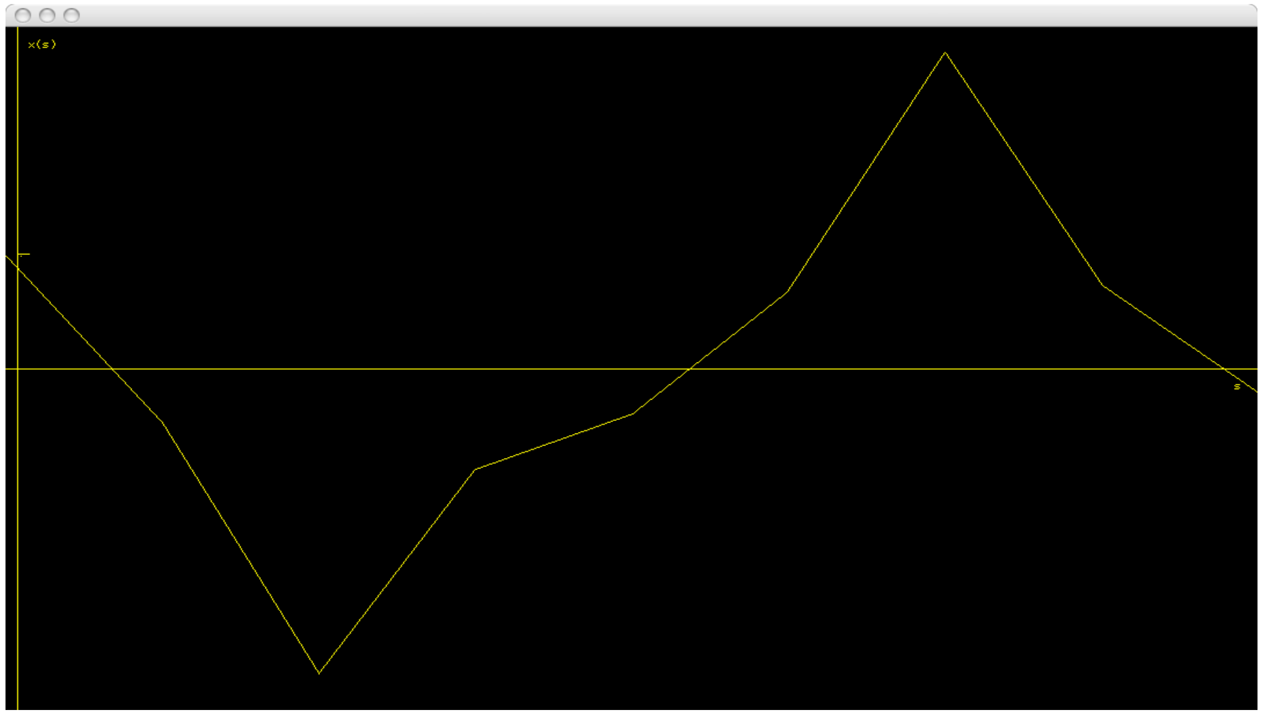


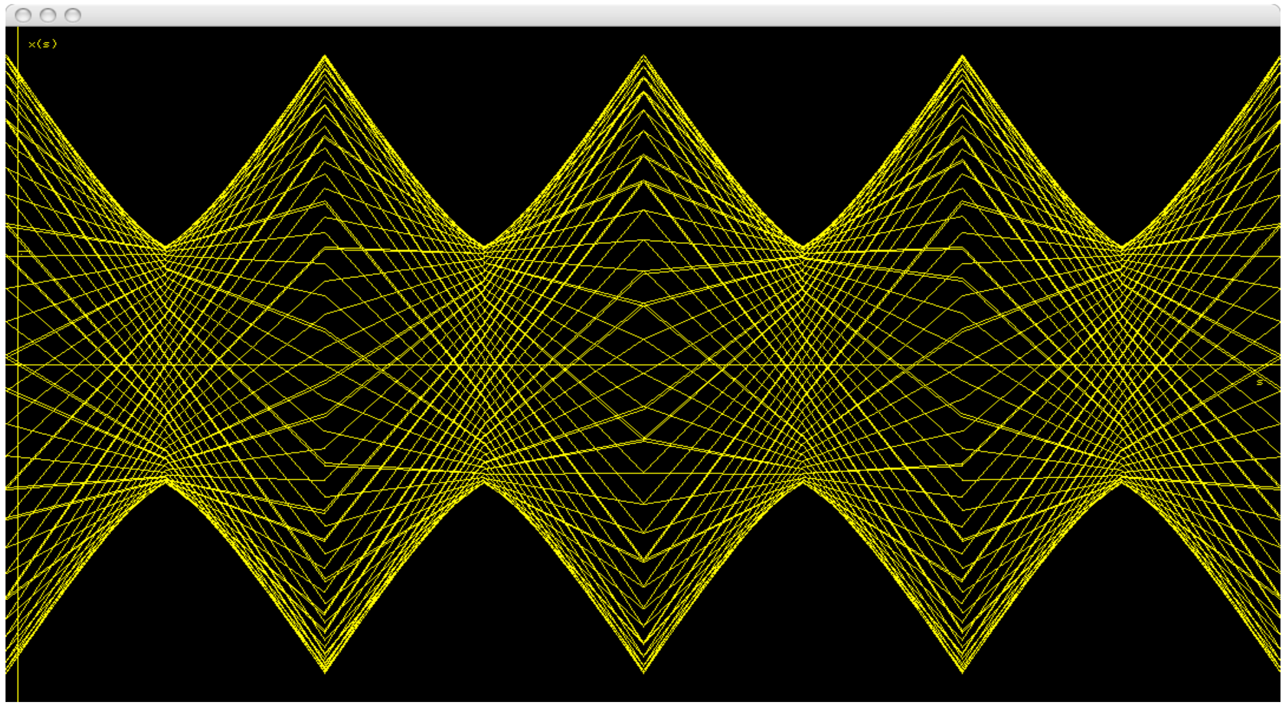


The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line





Can we describe the maximum amplitude of particle excursions in analytical form?

of course! *coming up soon …*

Particle Beams and Phase Space

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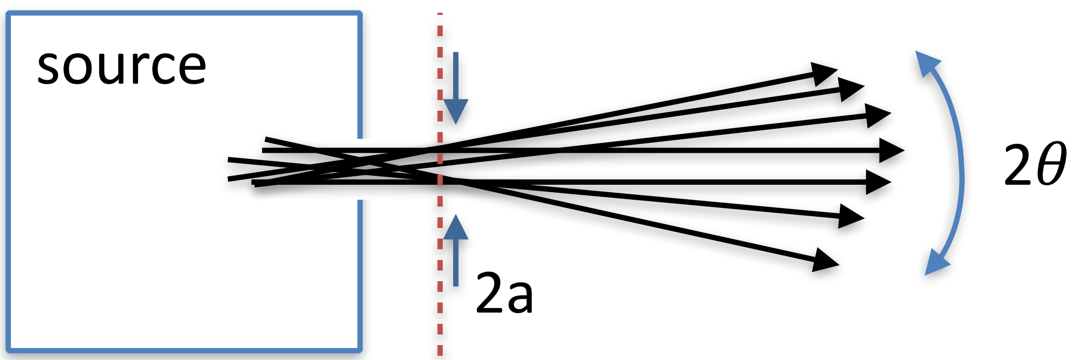
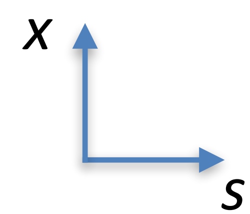
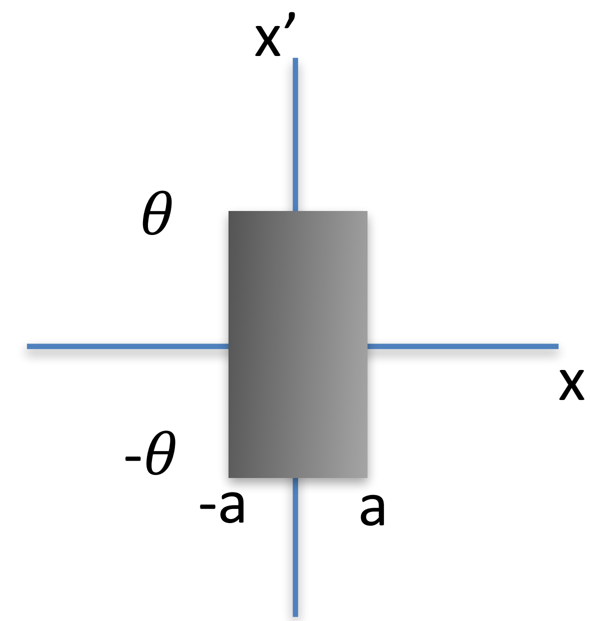
Transverse coordinates:



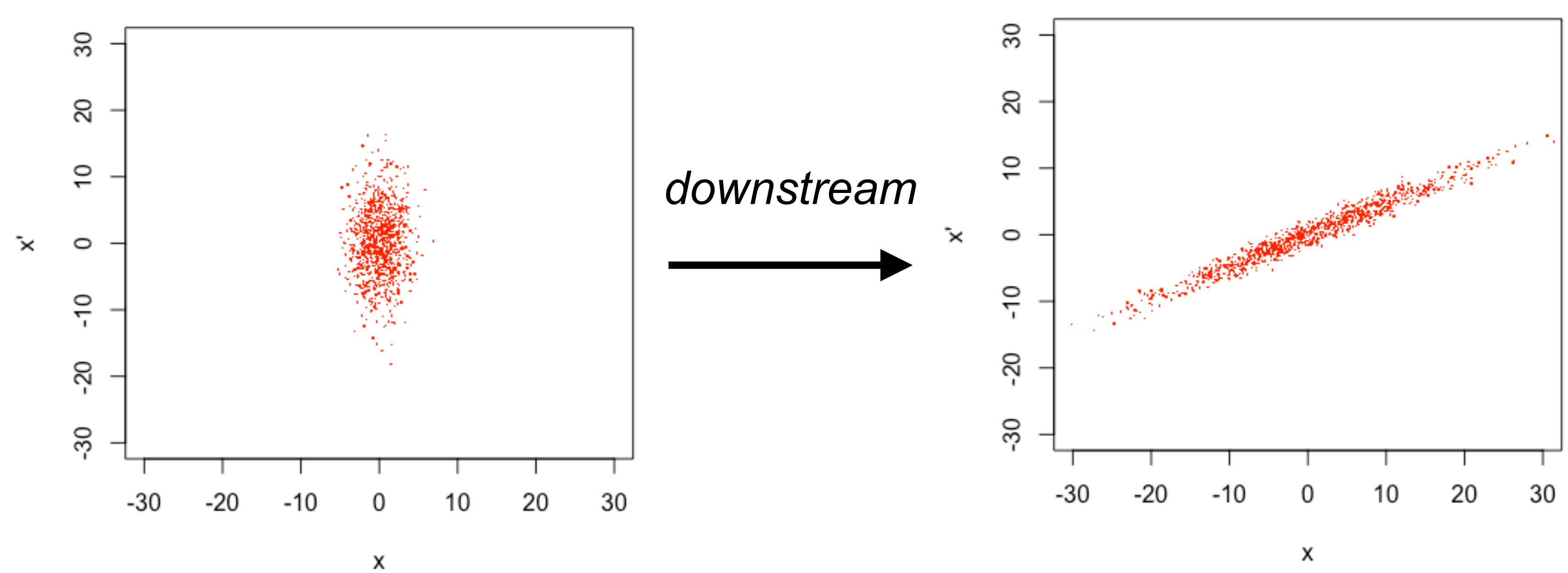


angle: $x' = dx/ds$

Phase Space:

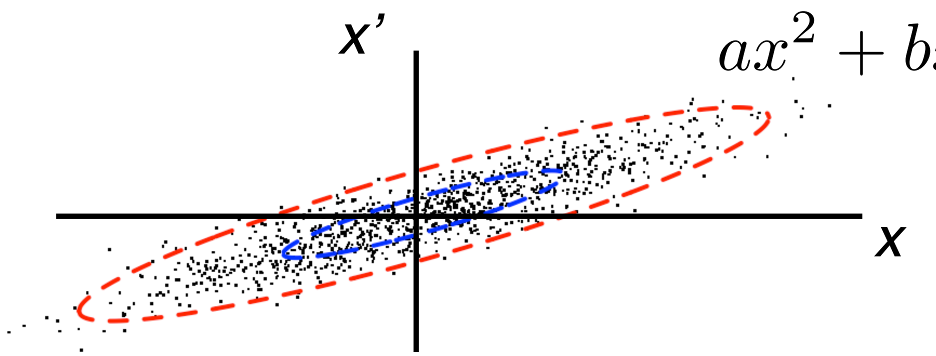


Shape, orientation of distribution in "phase space" will change as particles progress downstream, but effective "area" of distribution will remain constant (*Liouville*); correlations will naturally develop



Emittance in Terms of Moments

- Considering the general equation of an ellipse, the area enclosed by the ellipse is related to its coefficients by:



$$ax^2 + bxx' + cx'^2 = 1$$

area of ellipse:

$$\mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

- Can define quantities scaled by an area, ϵ, of our elliptical distribution:

$$\alpha \equiv -\frac{\langle xx' \rangle}{\epsilon/\pi} \qquad \beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \qquad \gamma \equiv \frac{\langle x'^2 \rangle}{\epsilon/\pi}$$

$$\epsilon = \pi\sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

the "rms emittance"

α, β, γ collectively are called the ***Courant-Snyder* parameters**, or *Twiss* parameters

So, equation of the **blue** curve above:

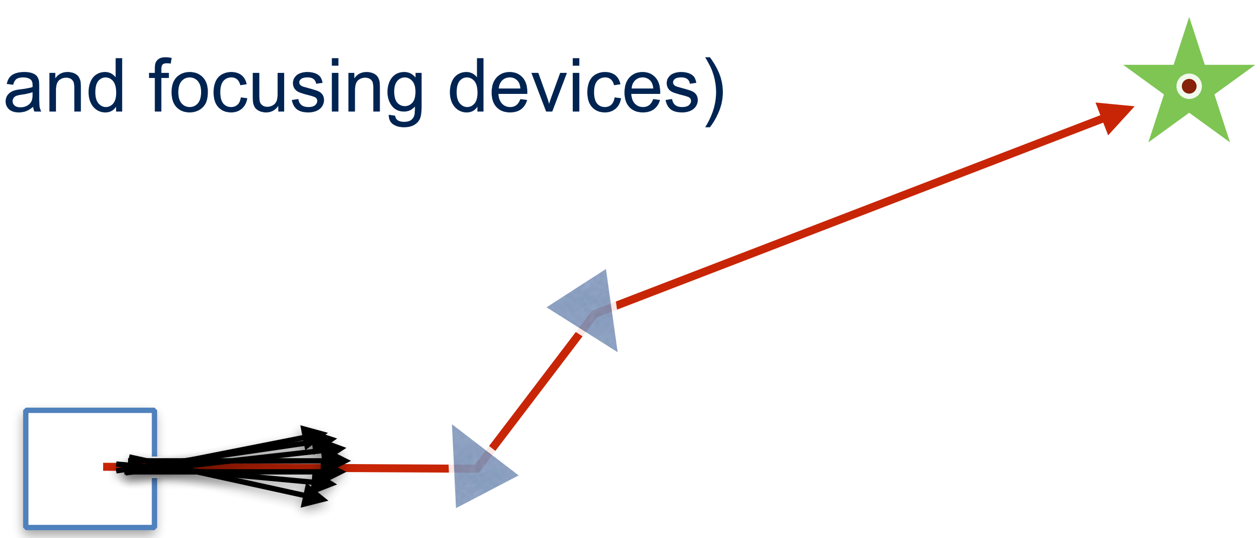
$$\gamma x^2 + 2\alpha xx' + \beta x'^2 = \epsilon/\pi$$

The ellipse (red curve above) that contains ~95% has area ~6ϵ

(*for Gaussian distribution*)

Essential Beam Transport and Focusing

- Can imagine using a section of finite length containing pure uniform magnetic field to bend a charged particle's trajectory through a portion of a circular arc, thus steering it in a new direction. An arrangement of such magnets can thus be used to guide an "ideal" particle from one point to another
- However, most (all?) particles are NOT ideal! Hence, as particles drift away from the ideal trajectory, we wish to guide them (using quadrupole magnets or solenoids) back toward the ideal.
- Will use discrete electromagnets of finite length and assume a linear relationship between a particle's exit trajectory to its entrance trajectory, depending upon the strength of the magnetic field
  - (similar rules for electrostatic bending and focusing devices)



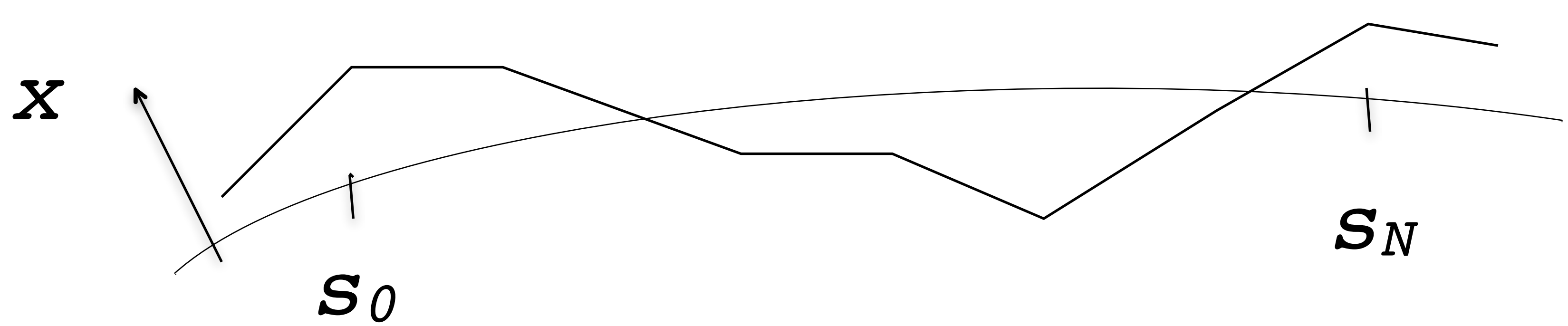
Linear Optics

- Let *x* be the transverse (horizontal, say) displacement of a particle from the ideal beam trajectory. Let the angle it makes to the ideal trajectory be *x'* = *dx*/*ds*, where *s* is the distance along the ideal trajectory. Transport through a magnetic element is then described by a matrix *M*, such that

$$\vec{X} = M\vec{X}_0 \qquad\qquad \vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

- An arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication for elements all along the beam line…

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



TRANSPORT of Beam Moments

- Transport of particle state vector downstream from position 0

$$\vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix} \qquad \vec{X} = M\vec{X}_0$$

- Create a "covariance matrix" of the resulting vector…

$$\vec{X}\vec{X}^T = \begin{pmatrix} x^2 & xx' \\ x'x & x'^2 \end{pmatrix} = M\vec{X}_0(M\vec{X}_0)^T = M\vec{X}_0\vec{X}_0^T M^T$$

- … then, by averaging over all the particles in the distribution,

$$\Sigma = \begin{pmatrix} \langle x^2\rangle & \langle xx'\rangle \\ \langle x'x\rangle & \langle x'^2\rangle \end{pmatrix}$$

we get: $$\Sigma = M\Sigma_0 M^T$$

TRANSPORT of Beam Moments

- So, since

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} \sim \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = \epsilon \cdot K$$

- where

$$K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$K = M \; K_0 \; M^T$$

- If know matrices *M*, then can “transport” beam parameters from one point to any point downstream, which determines beam distribution along the way.

$$\beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \quad \longrightarrow \quad x_{rms}(s) = \sqrt{\epsilon\beta(s)/\pi}$$

Conservation of Emittance

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- Note that from

$$\Sigma = M\Sigma_0 M^T$$

$$\Sigma = \epsilon \cdot K \qquad K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\det \Sigma = \det M \;\det \Sigma_0 \;\det M^T = \det \Sigma_0$$

- and

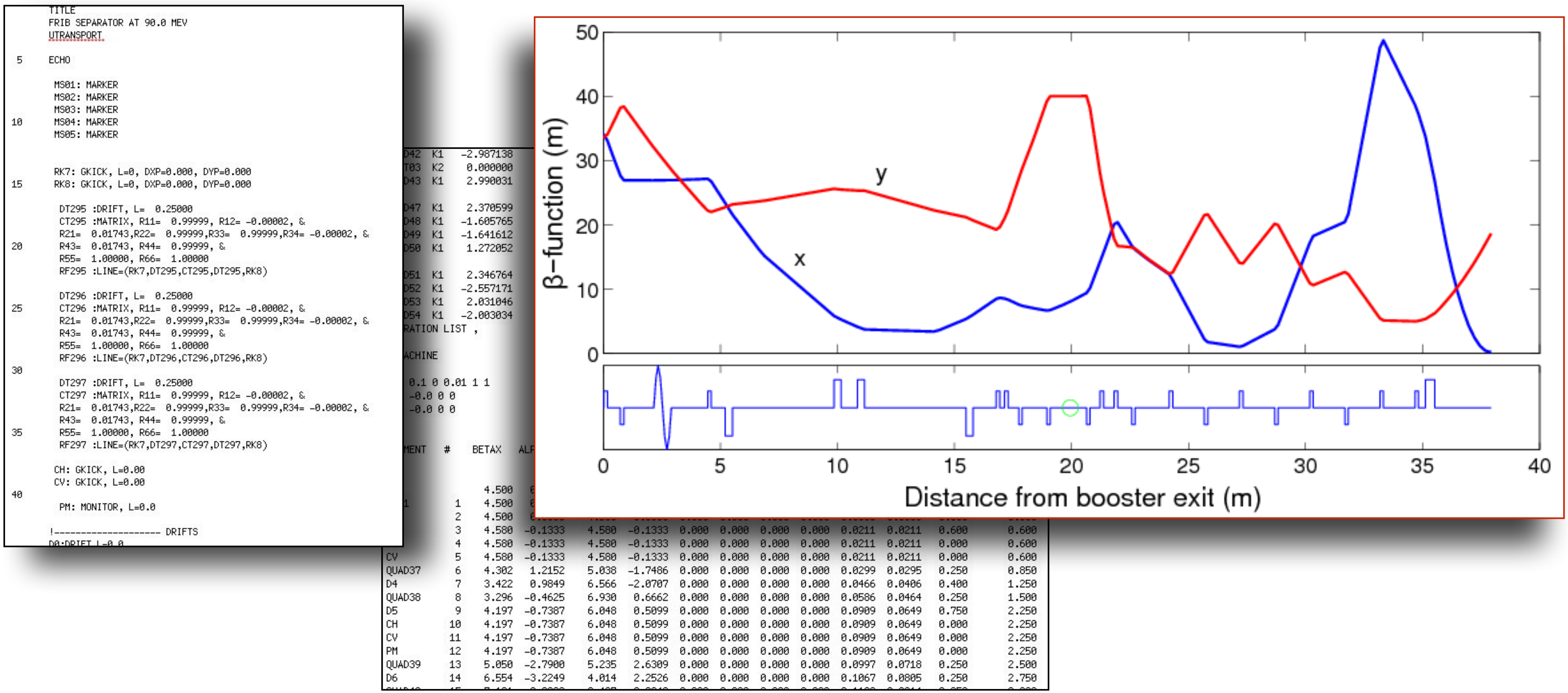
note: det*M* = 1

$$\det \Sigma = \epsilon^2 \det K = \epsilon^2(\beta\gamma - \alpha^2) = \epsilon^2$$

- Thus, the emittance is conserved upon transport through the system

Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT, MAD, DIMAD, TRACE, TRACE3D, COSY, SYNCH, CHEF, many more ...



Let's Think About the Numbers & Units…

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$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

- If <x^2> ~ mm^2, and <x'^2> ~ mrad^2, then the emittance can have units of mm-mrad (*also* = μm)
- Courant-Snyder parameters

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon}$$

mm^2/(mm-mrad) ~ mm/mrad = m

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

(mm-mrad)/(mm-mrad) = dimensionless

$$\gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

mrad^2/(mm-mrad) ~ 1/m

The "π" comes from our definition of emittance as an area in phase space; emittance is often expressed in units of "π mm-mrad"

Summary

- Given an initial particle distribution in phase space at the input to a beam transport system, can describe that distribution (sometimes not all that well, but we try…) using Courant-Snyder parameters:

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon} \qquad \gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

- The C-S parameters can then be computed downstream, using

$$\Sigma = M\Sigma_0 M^T$$