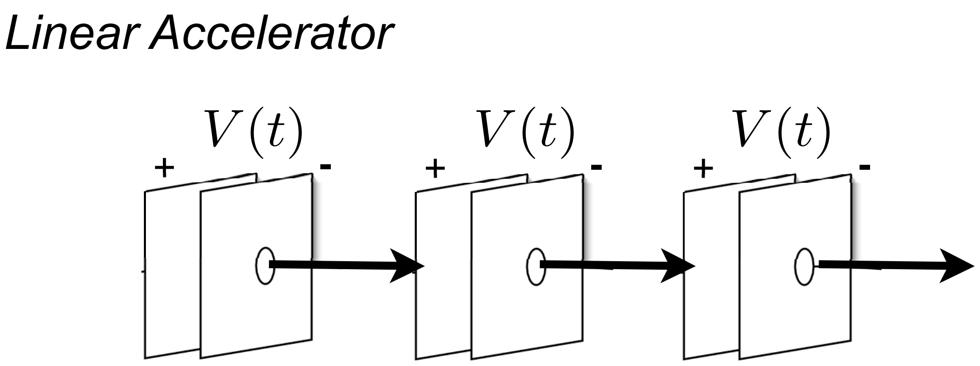
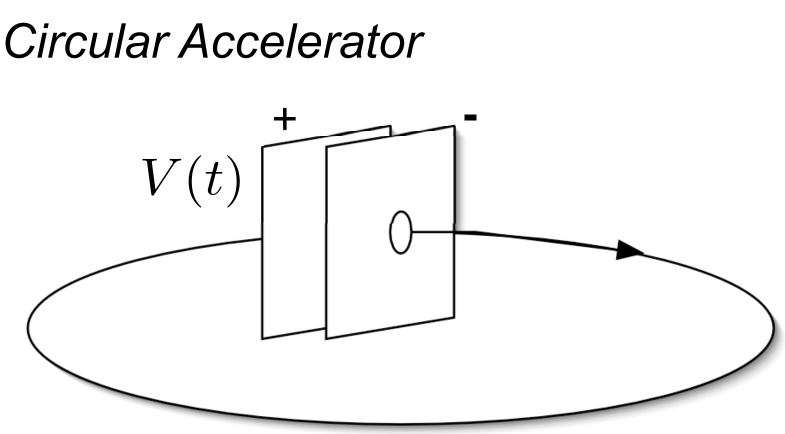
Linacs and Synchrotrons

- Essential difference:



- pass *N* cavities 1 time each

— or —



- pass 1 cavity *N* times

- otherwise, essentially the same longitudinal dynamics

Linacs and Synchrotrons

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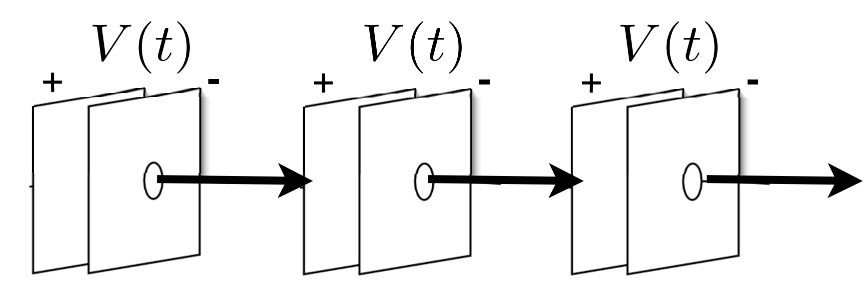
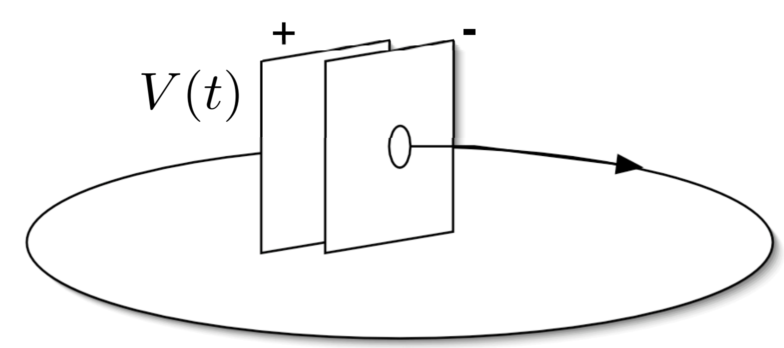
- Linac cavities can have different frequencies, each at different phases (e.g., FRIB); but typically one frequency, at least for major sections of the linac
- Synchrotron — with only 1 cavity system, — inherently same frequency, though its value must change if particle speed changes during acceleration (protons, ions)
- Must consider time of flight between cavities / passages

Linear Accelerator



Circular Accelerator



Repetitive Systems of Acceleration

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- We will assume that particles are propagating through a system of accelerating cavities. Each cavity has oscillating fields with frequency f_{RF}, and maximum "applied" voltage V (i.e., this takes into account TTF's, etc.). The ideal particle would arrive at the cavity at phase ϕ_s.
- We will choose ϕ_s to be relative to the "positive zero-crossing" of the RF wave, such that the ideal particle acquires an energy gain of

$$\Delta E_s = \Delta W_s = qV \sin \phi_s$$

» this definition used for synchrotrons; linacs more often define ϕ_s relative to the "crest" of the RF wave

- apologies for this possible *further* confusion…
  - the physics, of course, is the same

Acceleration of Ideal Particle

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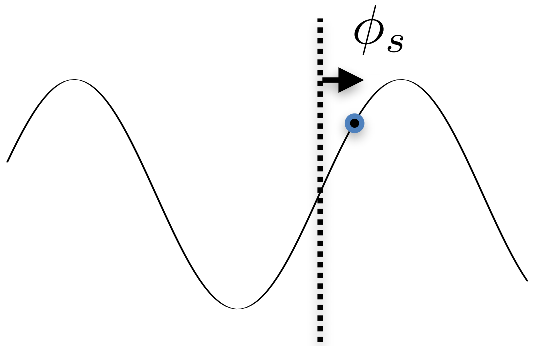
Wish to accelerate the ideal particle. As this particle exits the *(n+1)-th* RF cavity/station we would have



$$E_s^{(n+1)} = E_s^{(n)} + QeV \sin\phi_s$$

If we are considering a synchrotron, we can consider the above as the total energy gain on the *(n+1)-th* revolution. The ideal energy gain per second would be:

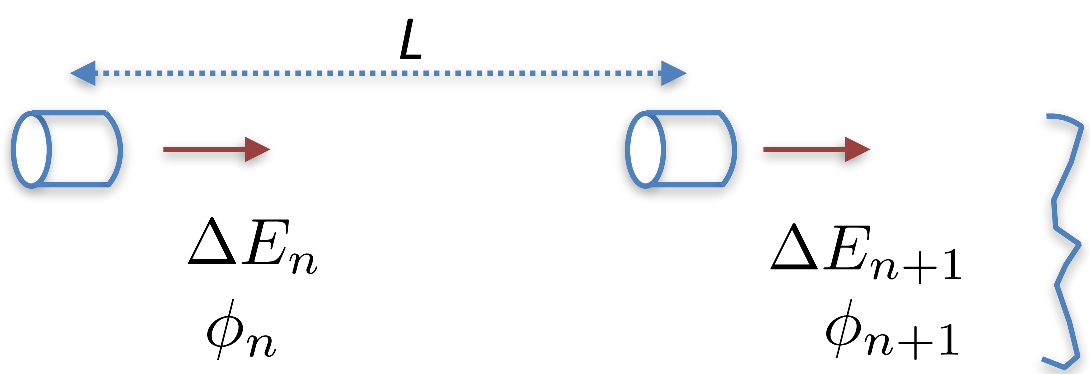
$$dE_s/dt = f_0 QeV \sin\phi_s$$

$f_0 = \text{revolution frequency}$

Next, look at (longitudinal) motion of particles near the ideal particle: ϕ = phase w.r.t. RF system

$\Delta E \equiv E - E_s$ = energy difference from the ideal

- Assume accelerating system of cavities is set up such that ideal particle arrives at each cavity when the accelerating voltage *V* is at the same phase (called the "synchronous phase"); consider a "test" particle:



$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

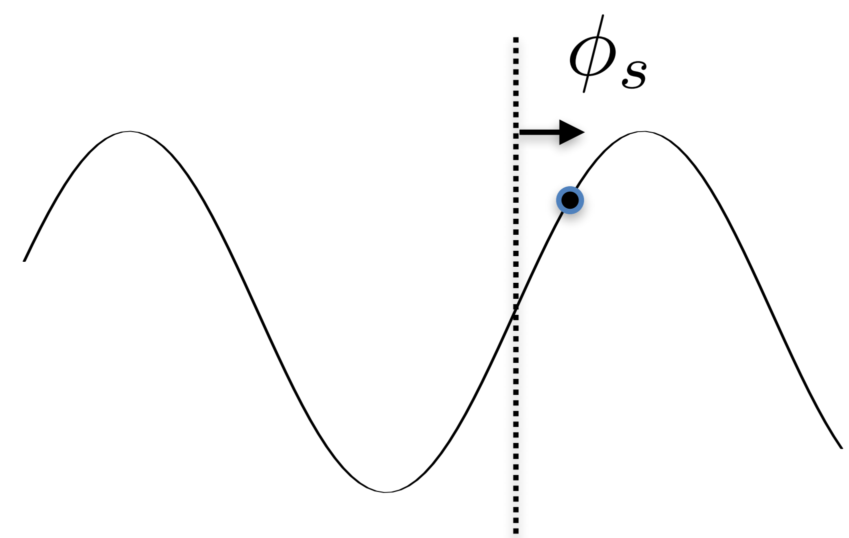
$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

(*difference equations*)

Notes:

$$h = L/\beta\lambda, \quad \lambda = c/f_{\rm rf} \quad \text{or,} \quad h = f_{\rm rf}L/v$$

Desire *h* to be an integer, to arrive at same phase each time.
If *L* is circumference of a synchrotron then: $h = f_{\rm rf}/f_0$
where f_0 is the revolution frequency,
In this case, *h* is called the "harmonic number"



$$E = mc^2 + W; \qquad \Delta E \Leftrightarrow \Delta W$$

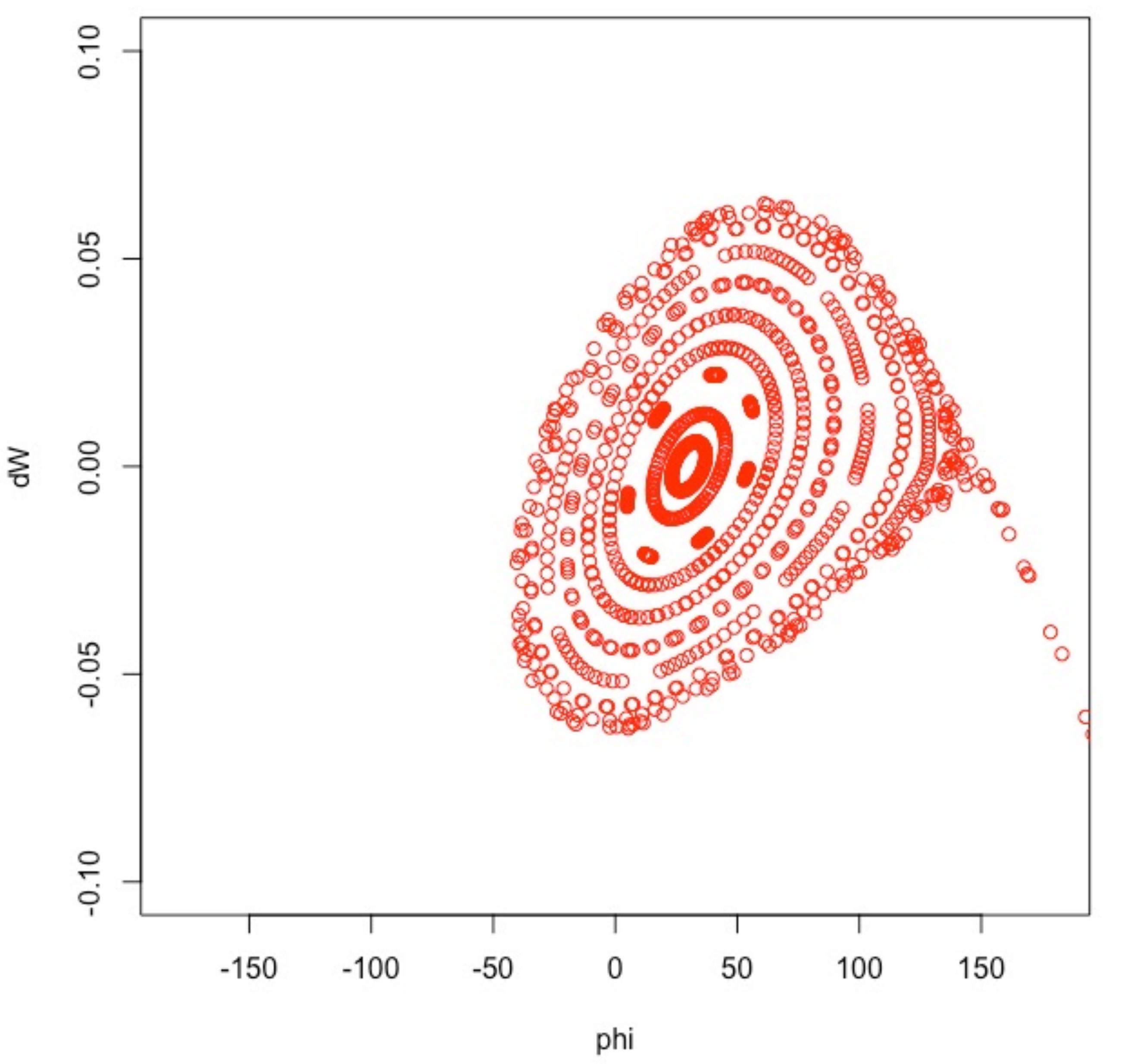
Applying the Difference Equations

```
while (i < Nturns+1) {
  phi = phi + k*dW
  dW  = dW + QonA*eV*(sin(phi)-sin(phis))
  points(phi*360/2/pi, dW, pch=21,col="red")
  i = i + 1
}
```

Let's run a code…

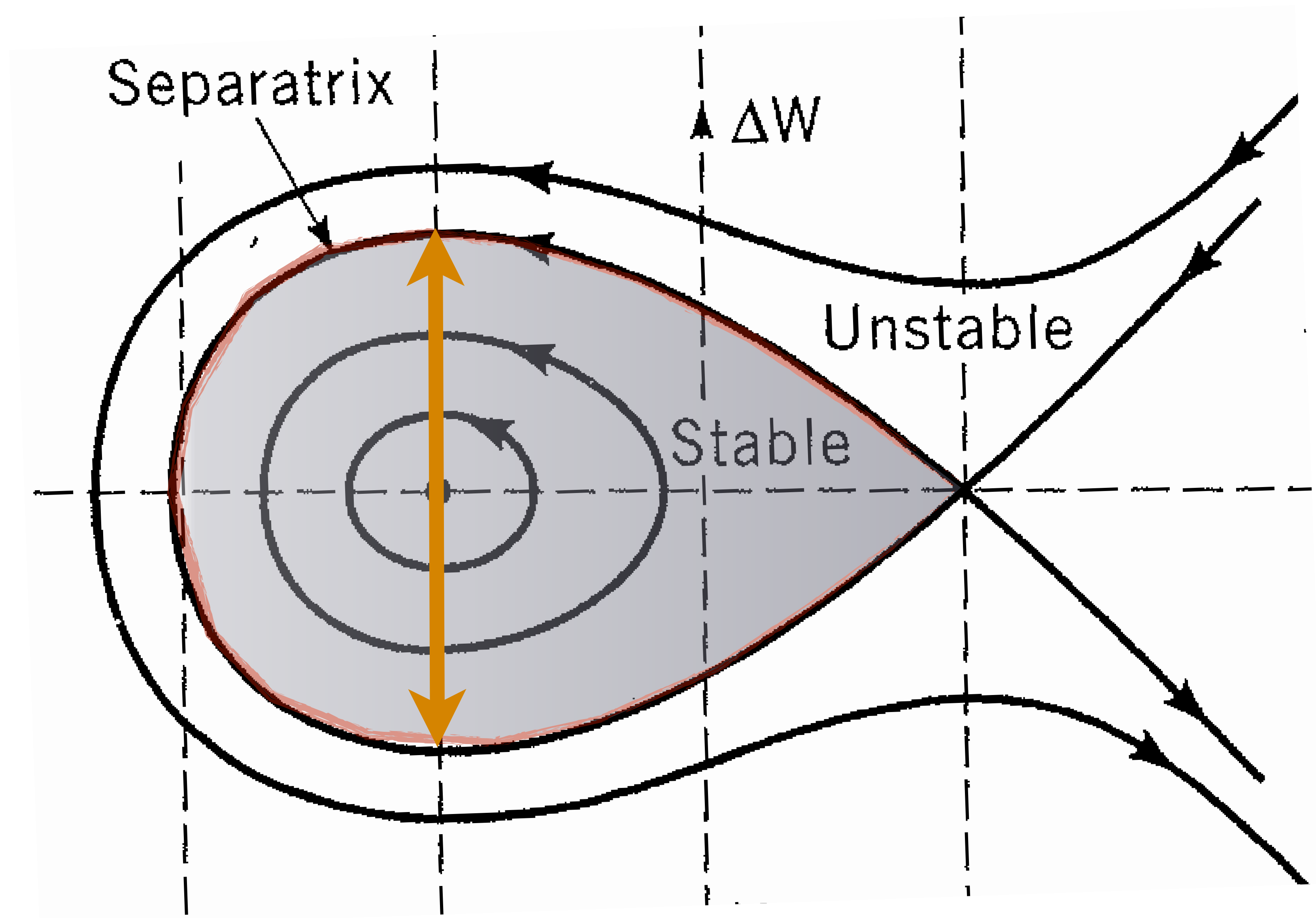
v0_RFtrack.R

```r
# Program to plot longitudinal phase space motion
#  through a system of cavities (just an example...)

Nturns = 100

# Some Parameters
Ws = 1.0 # MeV/u
phis = 30*pi/180  # synchronous phase angle
eV = 0.2 # MeV/u
QonA = 0.25
gamma = (931+Ws)/931
beta = sqrt(1-1/gamma^2)
eta = -1/gamma^2
h = 1/(beta*3e8/80.5e6)
k = 2*pi*h*eta/beta^2*(gamma-1)/gamma/Ws

# initialize the phase space plot
phi   = 0
dW    = 0
plot(phi, dW, xlim=c(-180,180), ylim=c(-0.1,0.1), typ="n")

trk = 1
while (trk < 16) {
# initialize particle positions in phase space
  u0 <- locator(1)
  phi   <- u0$x/180*pi
  dW    <- u0$y
# track the particle...
  i = 1
  while (i < Nturns+1) {
    phi = phi + k*dW
    dW  = dW + QonA*eV*(sin(phi)-sin(phis))
    points(phi*360/2/pi, dW, pch=21,col="red")
    i = i + 1
  }
trk = trk + 1
}
```



Acceptance and Emittance

- Stable region often called an RF “bucket”
  - “contains” the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system
-



Acceptance and Emittance

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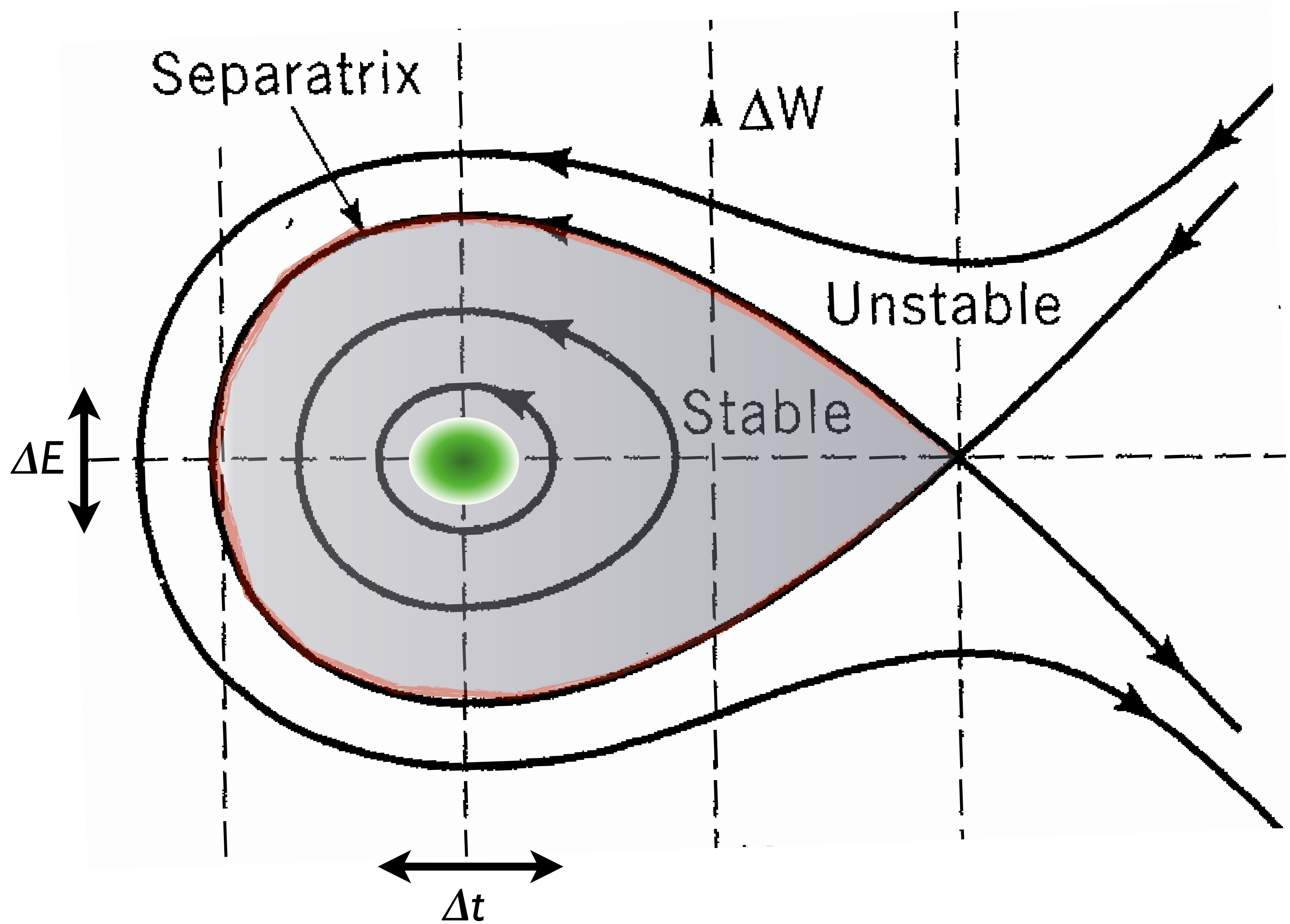
- Stable region often called an RF "bucket"
  - "contains" the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system
- Desire the beam particles to occupy much smaller area in the phase space



area: "eV-sec"

Note: *E*, *t canonical*

- got to here…

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

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differential approach...

start with above difference eqs $\rightarrow \dfrac{d\phi}{dn} = \dfrac{2\pi h\eta}{\beta^2 E}\,\Delta E, \qquad \dfrac{d\Delta E}{dn} = QeV(\sin\phi - \sin\phi_s)$

(1)

(2)

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

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$$\rightarrow \frac{d^2\phi}{dn^2} = \frac{2\pi h\eta}{\beta^2 E}\frac{d\Delta E}{dn} = \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) \qquad (1)$$

(2)

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

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$$\Rightarrow \frac{d^2\phi}{dn^2} - \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) = 0$$

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$$\Rightarrow \frac{d^2\phi}{dn^2} - \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) = 0$$

find 1st integral:

$$\int\left(\frac{d^2\phi}{dn^2}\right)\frac{d\phi}{dn}\,dn - \frac{2\pi h\eta}{\beta^2 E}QeV\int(\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn = 0$$

(2)

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

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find 1^{st} integral:

$$\int\left(\frac{d^2\phi}{dn^2}\right)\frac{d\phi}{dn}\,dn - \frac{2\pi h\eta}{\beta^2 E}QeV\int(\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn = 0$$

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{2\pi h\eta}{\beta^2 E}QeV(\cos\phi + \phi\sin\phi_s) = constant$$

(2)

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

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differential approach…

start with above difference eqs $\rightarrow \frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E}\Delta E, \qquad \frac{d\Delta E}{dn} = QeV(\sin\phi - \sin\phi_s)$

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find 1st integral:

$$\int \left(\frac{d^2\phi}{dn^2}\right)\frac{d\phi}{dn}\,dn - \frac{2\pi h\eta}{\beta^2 E}QeV\int(\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn = 0$$

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{2\pi h\eta}{\beta^2 E}QeV(\cos\phi + \phi\sin\phi_s) = constant$$

or,

$$\boxed{\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta}QeV(\cos\phi + \phi\sin\phi_s) = constant} \qquad (2)$$

The equation of the ***trajectories*** in phase space!

Synchrotron Oscillations

- Particles near the synchronous phase and ideal energy will oscillate about the synchronous particle with the "synchrotron frequency" (this is called *synchrotron motion*, even for a linac!) In a synchrotron, …
  - "synchrotron tune" == # of synch. osc.'s per revolution

compute small oscillation frequency:

$$\phi = \phi_s + \Delta\phi \quad \rightarrow \quad \sin\phi - \sin\phi_s = \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi - \sin\phi_s$$

$$\approx \Delta\phi \ \cos\phi_s$$

in (1), let

$$\Rightarrow \frac{d^2\Delta\phi}{dn^2} - \left(\frac{2\pi h\eta}{\beta^2 E} QeV \cos\phi_s\right) \Delta\phi = 0$$

$$(2\pi\nu_s)^2$$

$$\Rightarrow \quad \nu_s = \sqrt{-\frac{h\eta QeV}{2\pi\beta^2 E}\cos\phi_s}$$

$$\text{if } \eta > 0, \text{ choose } \cos\phi_s < 0$$

Comment on Frequencies of the Motion

- From what we've just seen, the synchrotron motion in a circular accelerator takes many (perhaps hundreds of) revolutions to complete one synchrotron period
- On the other hand, in the transverse plane, a particle will typically undergo many betatron oscillations during one revolution
- Thus, transverse/longitudinal dynamics typically occur on very different time scales — this actually justifies us studying them independently

Motion Near the Ideal Particle

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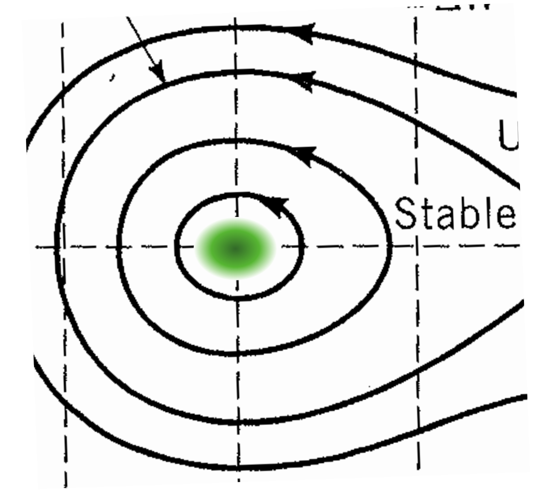
Linearize the motion, and write in matrix form…

$$\phi = \phi_s + \Delta\phi$$



$$\begin{aligned}
\phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s) \\
&= \Delta E_n + QeV(\sin\phi_s \cos\Delta\phi_{n+1} + \sin\Delta\phi_{n+1}\cos\phi_s) - \sin\phi_s) \\
&= \Delta E_n + QeV\cos\phi_s\,\Delta\phi_{n+1} \\
&= \Delta E_n + QeV\cos\phi_s\left[\Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta\phi_{n+1} &= \Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= QeV\cos\phi_s\Delta\phi_n + \left(1 + \frac{2\pi h\eta}{\beta^2 E}QeV\cos\phi_s\right)\Delta E_n
\end{aligned}$$

or,

$$\begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_{n+1} = \begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ QeV\cos\phi_s & \left(1 + \frac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s\right) \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_n$$

$$= \begin{pmatrix} 1 & 0 \\ QeV\cos\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_n$$

$M \quad = \quad M_c \quad \cdot \quad M_d$

"thin" cavity (*acts as longitudinal focusing element*)

drift

Note: for $\eta < 0$, M_d is a "backwards" drift; i.e., $\Delta\phi$ decreases for ΔE>0

(when no bending)

$\eta = -1/\gamma^2$ in straight region (linac)

Remember from transverse motion, $x \propto \sqrt{\beta} \sin \Delta\psi$
and when *M* was periodic,

$$M = \begin{pmatrix} \cos\Delta\psi + \alpha \sin\Delta\psi & \beta \sin\Delta\psi \\ -\gamma \sin\Delta\psi & \cos\Delta\psi - \alpha \sin\Delta\psi \end{pmatrix} \qquad \text{and} \quad trM = 2\cos\Delta\psi$$

$\Delta\psi$ = phase advance through periodic section

Can imagine "longitudinal" *β*, *α*, *γ*, *Δψ* parameters as well
Note: from *M* of previous page, if represents periodic structure (synchrotron or portion of linac), then

$$trM = 2 + \frac{2\pi h\eta}{\beta^2 E} QeV \cos\phi_s = 2\cos\Delta\psi_s \qquad \Delta\psi_s = 2\pi\nu_s$$

longitudinal phase advance

oscillation frequency
w.r.t. cavity number, "*n*"
(e.g., synchrotron *tune*)

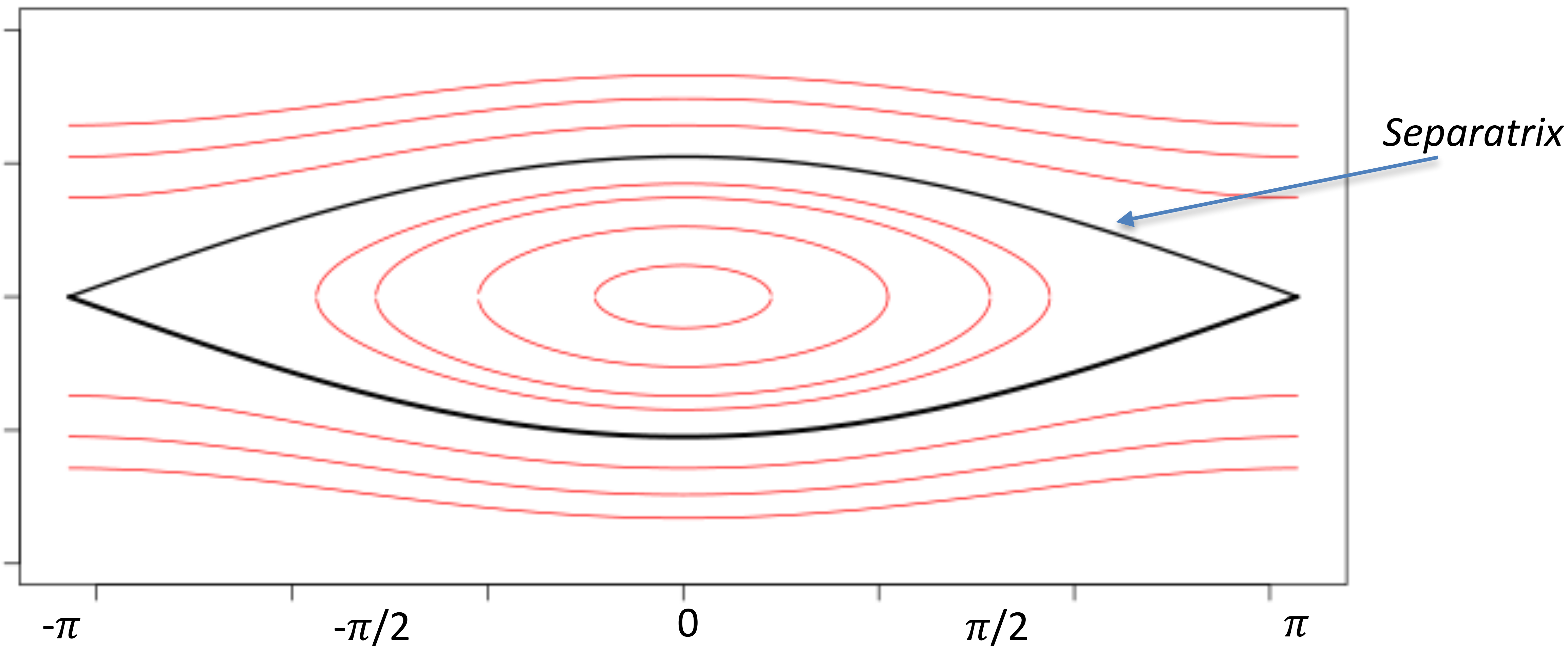
$$\cos\Delta\psi_s \approx 1 - \frac{1}{2}(\Delta\psi_s)^2 = 1 + \frac{\pi h\eta}{\beta^2 E} QeV \cos\phi_s \left[= \frac{1}{2} trM \right]$$

$$\Rightarrow \qquad \nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV \cos\phi_s}$$

as found previously!

The Stationary Bucket

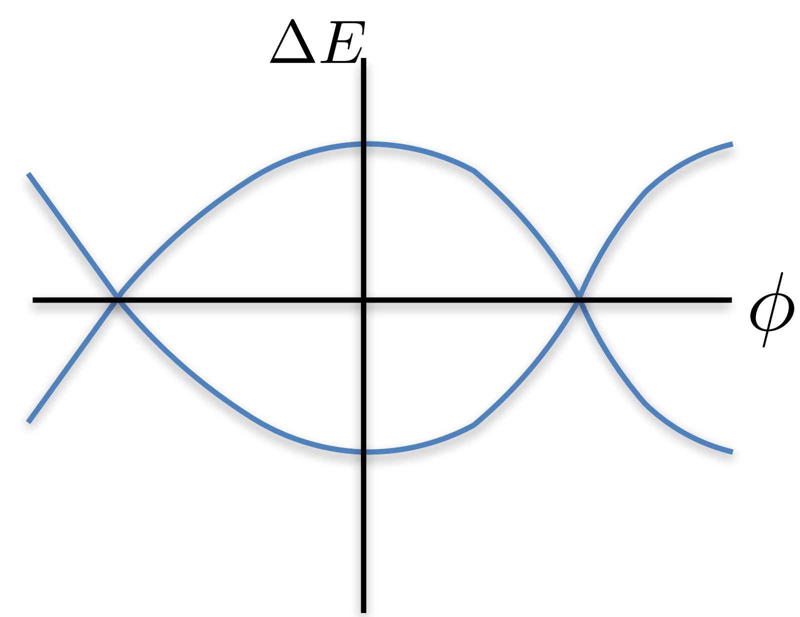
- Suppose do not wish to accelerate the ideal particle…
  - for lower energies, where the slip factor is negative, then need to choose $\phi_s = 0^o$



“stationary” bucket: $\phi_s = 0,\ 2\pi \quad (\sin\phi_s = 0)$ —> no average acceleration

anticipate stability: —> choose $\phi_s = 0, \quad \eta < 0$

then, $$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV \cos\phi = constant$$



on the separatrix: $\Delta E = 0 \ \text{ at } \ \phi = \pm\pi$

$$0 - 2\frac{\beta^2 E}{2\pi h\eta} QeV = constant$$

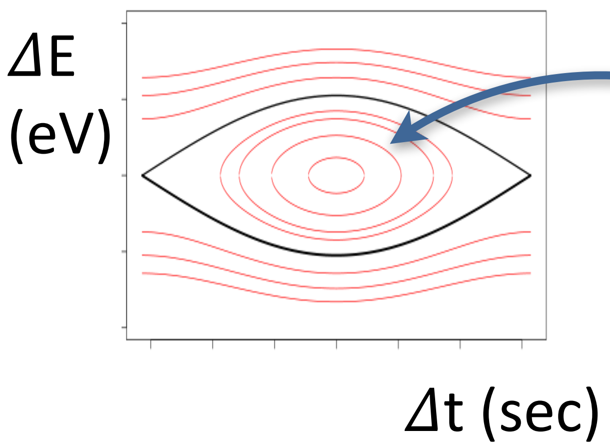
thus, the Eq. of separatrix: $$\Delta E^2 + (1+\cos\phi)\frac{\beta^2 E}{\pi h\eta} QeV = 0$$

$$\Delta E^2 + \frac{2\beta^2 E}{\pi h\eta} QeV \cos^2(\phi/2) = 0$$

separatrix: $$\Delta E = \pm\sqrt{-\frac{2\beta^2 E}{\pi h\eta} QeV} \cos(\phi/2)$$ *(for “stationary bucket”)* thus, "bucket height": $$a = \sqrt{\frac{2\beta^2 E}{\pi h|\eta|} QeV}$$

Phase space area of a stationary bucket: $$4\int_0^\pi a\ \cos(\phi/2)\ d\phi = 8a$$

and, if use *ΔE-Δt* coordinates rather than *ΔE-ϕ*, then area of a *stationary* bucket is…

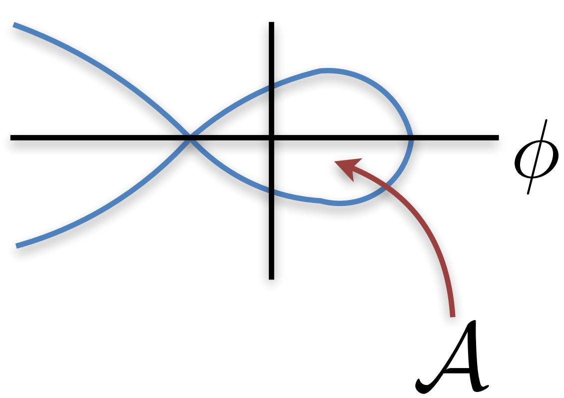


$$\mathcal{A}_0 \equiv \frac{8}{\pi f_{\rm rf}} \sqrt{\frac{\beta^2 EQeV}{2\pi h|\eta|}}$$

(*here, units of* eV-sec)

since $\phi = 2\pi f_{\rm rf} t$

Note: for $\sin\phi_s \neq 0$

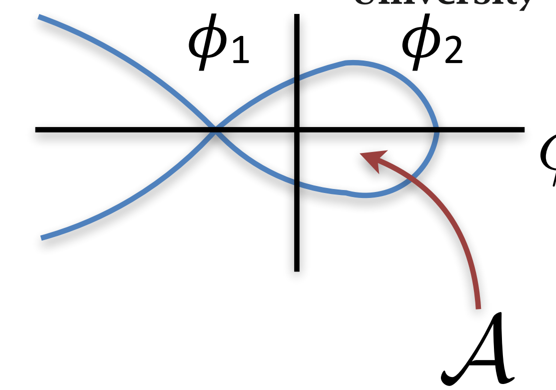


$$\mathcal{A} = \mathcal{A}_0 \cdot \mathcal{F}(\phi_s)$$

where $0 < \mathcal{F} < 1$

(determined numerically)

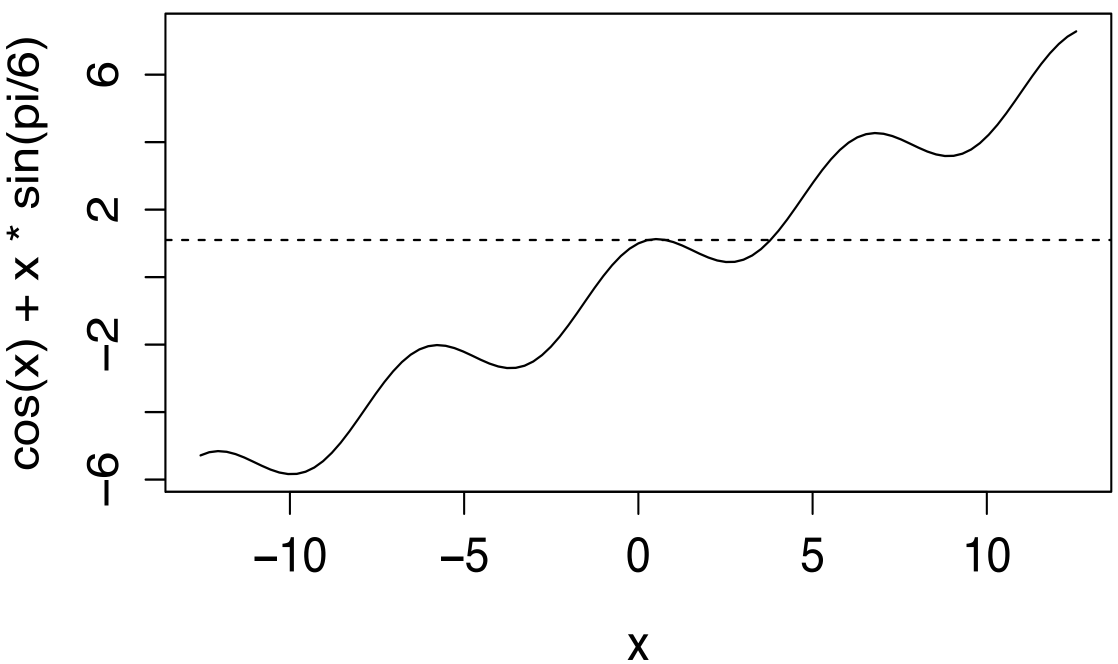
Area of a Moving Bucket



—> net average acceleration

curve: $$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta}QeV(\cos\phi + \phi\sin\phi_s) = constant$$

"kinetic"-like *"potential"-like* *"total Energy"-like*



ϕ_1 is where "potential like" has derivative = 0: $\phi_1 = \pi - \phi_s$

Given $\phi_1 = \pi - \phi_s$, can now determine the "constant": $\Delta E = 0$ at ϕ_1, and so…

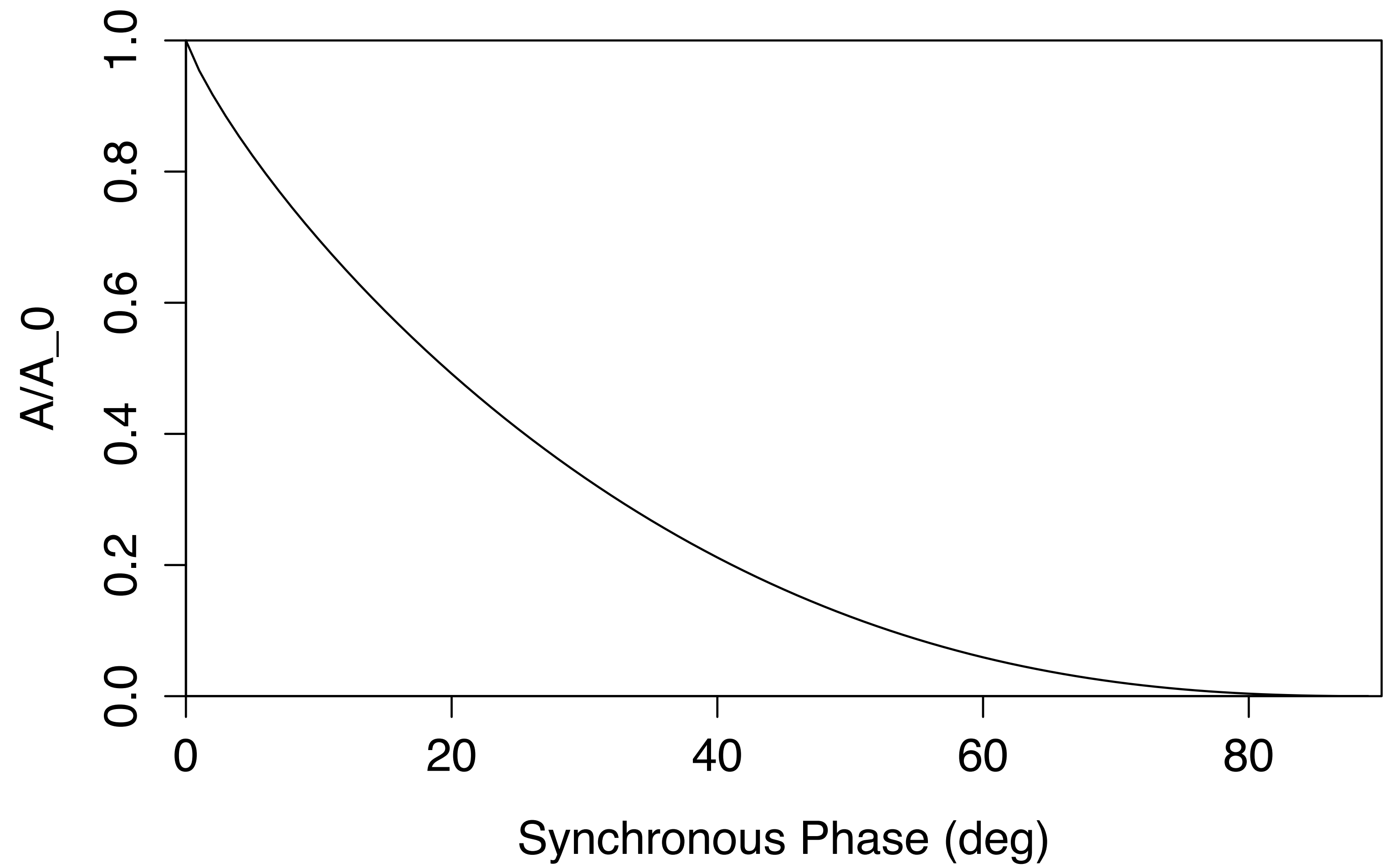
$$(0)^2 + 2\frac{\beta^2 E}{2\pi h\eta}QeV(\cos\phi_1 + \phi_1\sin\phi_s) = constant$$

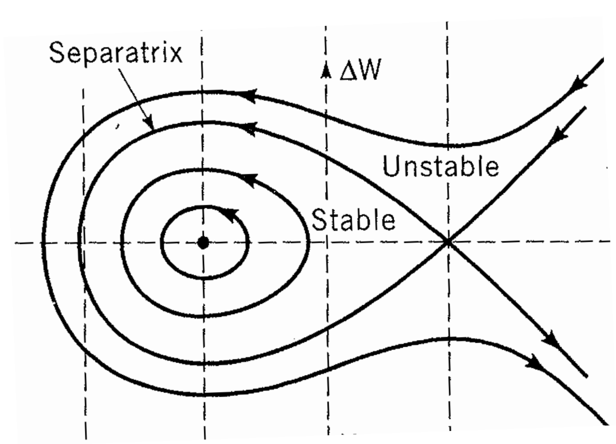
Then, find that ϕ_2 must satisfy:

$$\cos\phi_2 + \phi_2\sin\phi_s + \cos\phi_s + (\pi - \phi_s)\sin\phi_s = 0$$

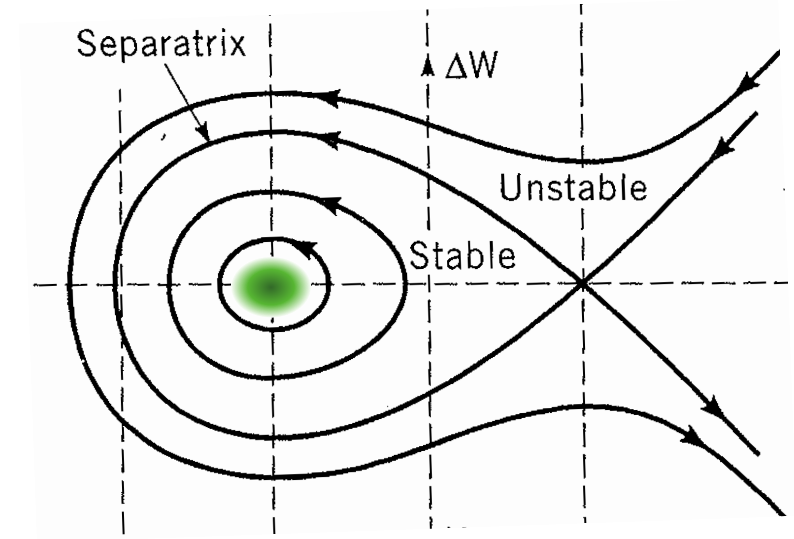
Numerical Solution for Bucket Area

```
# Solve for bucket area; phis = 0 is "stationary"

Xout <- array(0,dim=c(91,4))
phisDeg <- -1

for(i in (1:90)){
phisDeg <- phisDeg + 1
phis <- phisDeg*pi/180

f <- function(x){
   cos(x)+x*sin(phis)+cos(phis)-(pi-phis)*sin(phis) }
dE <- function(x){
   sqrt(cos(phis)-(pi-phis)*sin(phis)+cos(x)+x*sin(phis)) }

phi1 <- pi-phis
phi2 <- uniroot( f, c(-pi, 2*pi))$root
A <- -1/4/sqrt(2)*integrate(dE, phi1, phi2)$value

Xout[i,] = c(phis*180/pi, phi1*180/pi, phi2*180/pi, A)  }

plot(Xout[,1],Xout[,4],typ="l",
   xlab="Synchronous Phase (deg)", ylab="A/A_0",
   xaxs="i", yaxs="i",xlim=c(0,90))

Xout
```





Back to Small Oscillations…



from (2),

if $\phi = \phi_s + \Delta\phi$, then …
(*small*)

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi_s \cos\Delta\phi - \sin\phi_s \sin\Delta\phi + (\phi_s + \Delta\phi)\sin\phi_s) = constant$$

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi_s(1 - \frac{1}{2}\Delta\phi^2) - \sin\phi_s\Delta\phi$$
$$+\phi_s \sin\phi_s + \Delta\phi \sin\phi_s) = constant$$

$$\Delta E^2 + \left(-\frac{\beta^2 E}{2\pi h\eta} QeV \cos\phi_s\right)\Delta\phi^2 = constant \qquad (3)$$

This Eqn. represents trajectories in longitudinal phase space of particles ***near*** the ideal particle.

Back to Small Oscillations…

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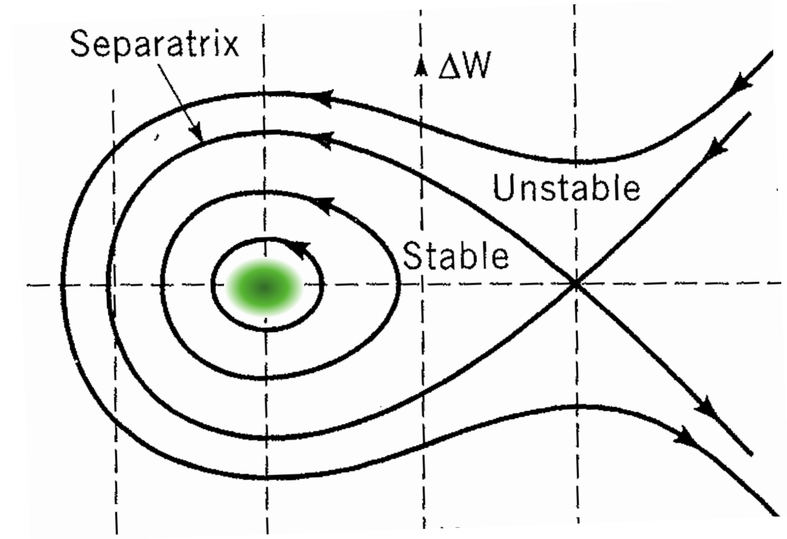
from (2), $$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi + \phi\sin\phi_s) = constant$$

if $\phi = \phi_s + \Delta\phi$, then …

(small)



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This Eqn. represents trajectories in longitudinal phase space of particles ***near*** the ideal particle.

Beam Longitudinal Emittance

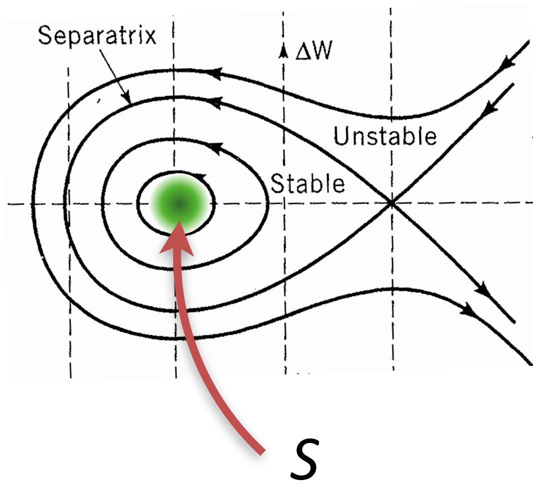
Northern Illinois University

Suppose beam is well contained within an ellipse given by (3), and suppose we know either $\Delta\hat{E}$ or $\Delta\hat{\phi}$ (or, $\Delta\hat{t}$) of the distribution (i.e., maximum extent). Then, the *constant* is easily seen to be:

$$constant = \Delta\hat{E}^2 = -\frac{\beta^2 E}{2\pi h\eta} QeV\cos\phi_s \Delta\hat{\phi}^2$$

So, area of ellipse (the *longitudinal emittance*) is: $\pi\ \Delta\hat{E}\Delta\hat{\phi}$

or, in *E-t* coordinates, $$S \equiv \pi\ \Delta\hat{E}\Delta\hat{t} = \pi\ \Delta\hat{E}\frac{\Delta\hat{\phi}}{2\pi f_{\rm rf}}$$



$$S = \frac{1}{2f_{\rm rf}}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta}Q\cos\phi_S}\ \ \Delta\hat{\phi}^2$$

or, $$S = 2\pi^2 f_{\rm rf}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta}Q\cos\phi_S}\ \ \Delta\hat{t}^2$$

units: "eV-sec"

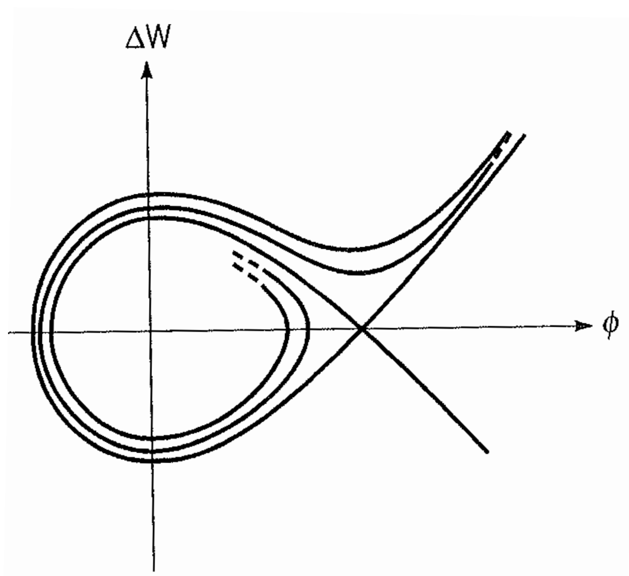
Golf Clubs *vs.* Fish

Northern Illinois University

- Our analysis "assumes" slowly changing variables (including the energy gain!). Quite reasonable in many Alvarez-style linacs and in synchrotrons
- In linacs, fractional energy change can be large, and so this will distort the phase space
- Plots from Wangler's book:

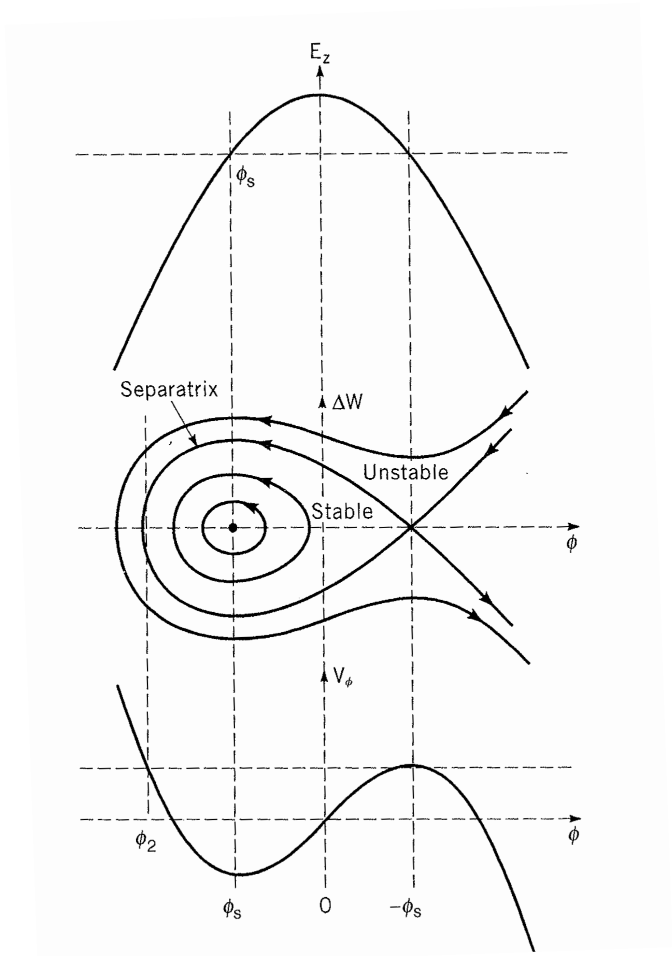
Here, a more rapid acceleration is included

(linac)



Here, assume that energy is "constant" or varying very slowly

(synchrotron)



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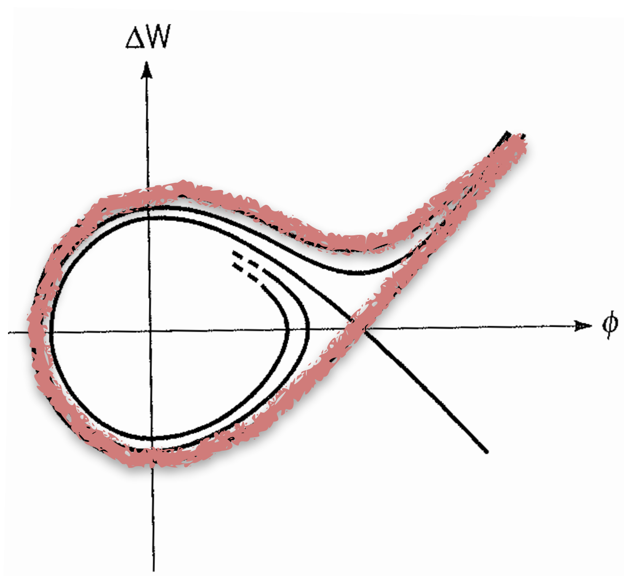

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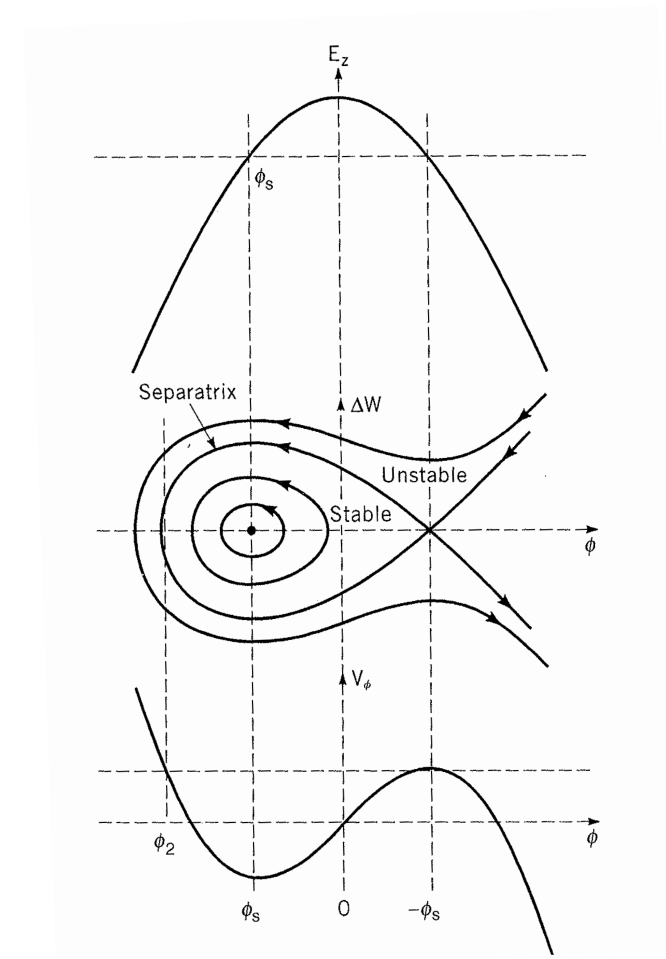

Momentum Compaction Factor

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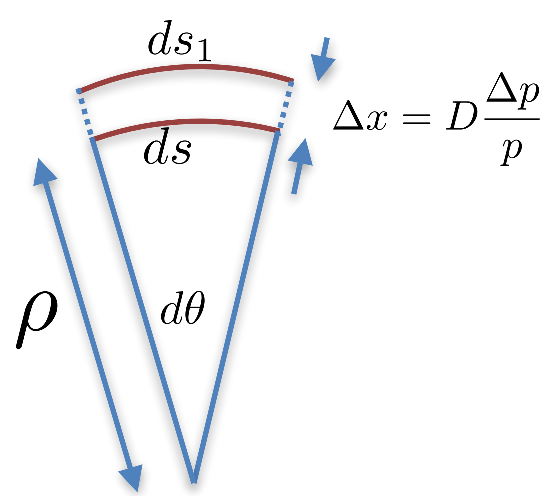
- How does path length along the beam line depend upon momentum?
  - in straight sections, no difference; in bending regions, *can* be different

Look closely at an infinitesimal section along the ideal trajectory…



$$d\theta = \frac{ds}{\rho} = \frac{ds_1}{\rho + \Delta x}$$

$$ds_1 - ds = \left(\frac{\rho + \Delta x}{\rho} - 1\right) ds$$

$$= \frac{\Delta x}{\rho} ds = \frac{D}{\rho}\,\frac{\Delta p}{p}\, ds$$

if L = path length along ideal trajectory between 2 points, then

$$\frac{\Delta L}{L} = \boxed{\frac{\int \frac{D(s)}{\rho(s)} ds}{\int ds}} \cdot \frac{\Delta p}{p}$$

The relative change in path length, per relative change in momentum, is called the *momentum compaction factor,*

$\alpha_p = \langle D/\rho \rangle$ along the ideal path

Transition Energy

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- In a synchrotron, there can be an energy at which the slip factor changes sign — this is call the “transition energy”

$$\eta = \alpha_p - \frac{1}{\gamma^2} = \left\langle \frac{D}{\rho} \right\rangle - \frac{1}{\gamma^2}$$

$$\eta = 0 = \alpha_p - \frac{1}{\gamma^2}$$

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

$$\boxed{\gamma_t \equiv \frac{1}{\sqrt{\alpha_p}}}$$

- In a typical FODO-style synchrotron, the transition gamma is roughly equal to the betatron tune

Transition

Northern Illinois University

We had... $\Rightarrow \frac{d^2\Delta\phi}{dn^2} - \left(\frac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s\right)\Delta\phi = 0$

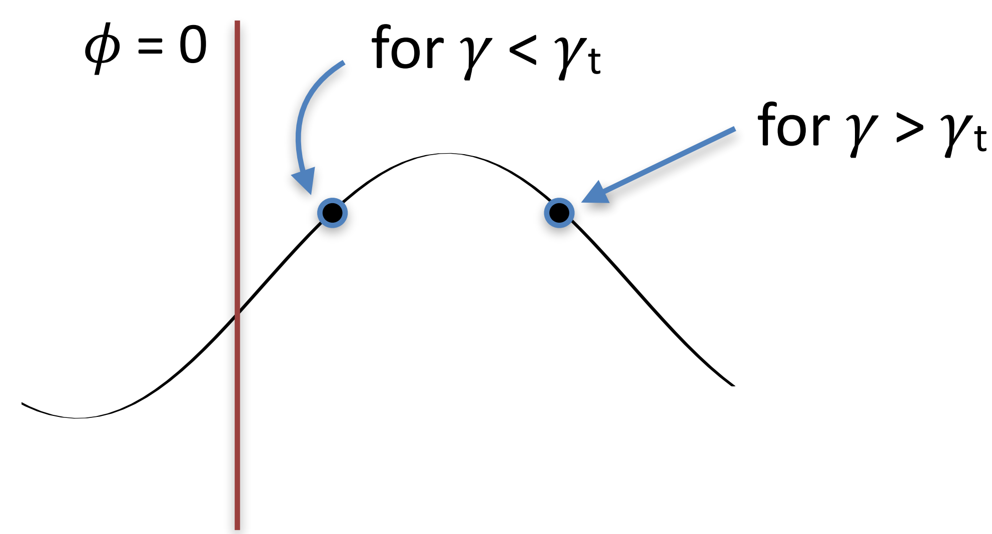
$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV\cos\phi_s}$$

$\text{if } \eta > 0, \text{ choose } \cos\phi_s < 0$

So,

when $\eta < 0$, we want $\cos\phi_s > 0$

when $\eta > 0$, we want $\cos\phi_s < 0$

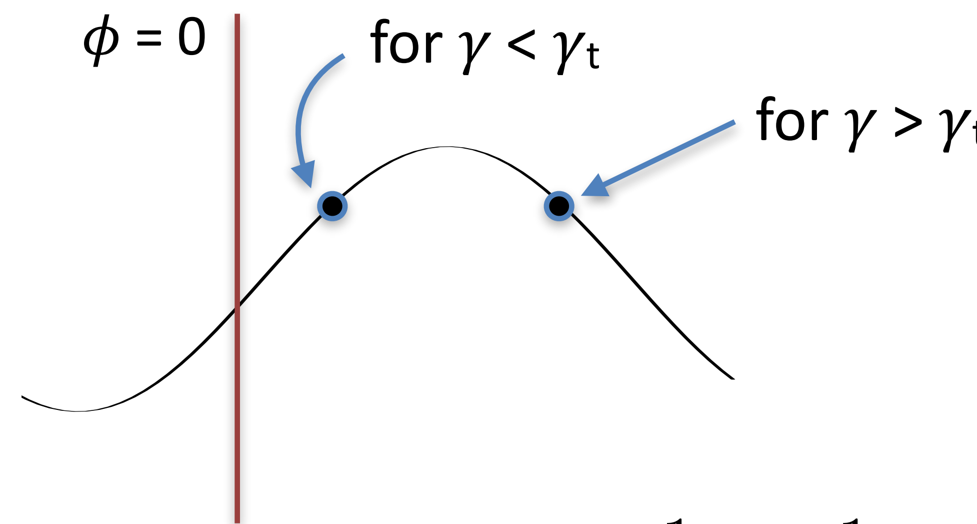


$\therefore$ if γ_t exists, need "phase jump" to occur at transition crossing

$$\gamma_t mc^2 = \text{transition energy}$$

Transition Crossing

- If the synchrotron accelerates through its transition energy, then the phase of the RF system has to be shifted at the time of transition crossing
- The synchrotron motion slows down as approach transition — it would stop if the slip factor were exactly zero!
  - loss of phase stability!
  - momentum spread also gets larger near transition
- So, best to accelerate quickly through this energy region!



$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV \cos\phi_s}$$

Buckets, Bunches, Batches, …

- Have seen definition of "buckets" — stable phase space area
- Buckets can be occupied by "bunches" of particles
  - note: need not be — can have "empty buckets"
  - thus, can (in principle) adjust bunch spacing, bunch arrangements, etc.
- A set of bunches that are created in an accelerator (pulsed) is often called a Batch (especially if from a synchrotron)
  - can also be called a Bunch Train as well (especially if from a linac)

Some Movies…

- Bucket Transformation
- Snap Capture
- Adiabatic Capture
- Parabolic acceleration
- Parabolic acceleration — full bucket
- Transition Crossing

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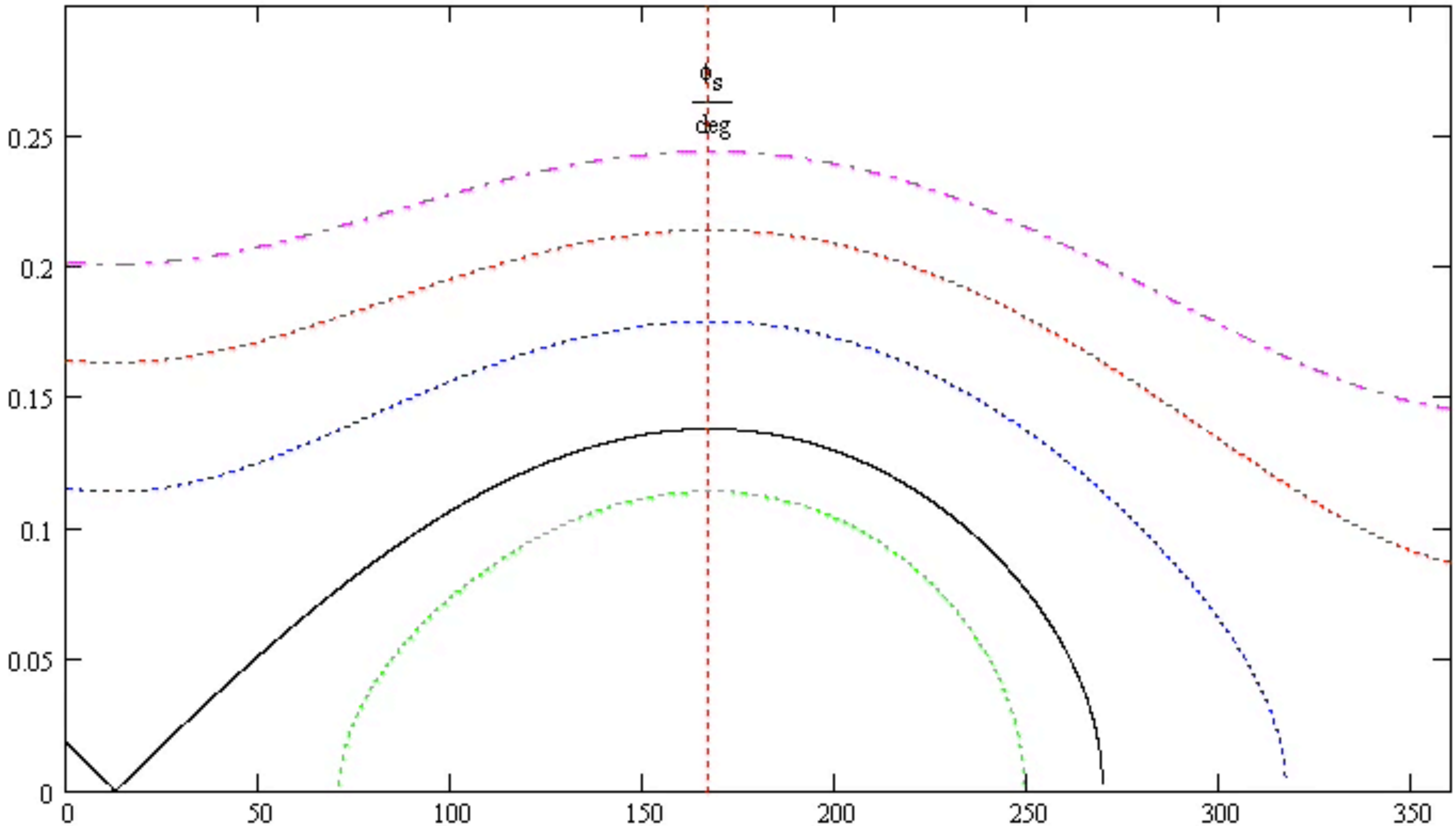
Phase space contours, for various values of k. Synchronous phase: $\phi_s = 167.25\,\text{deg}$



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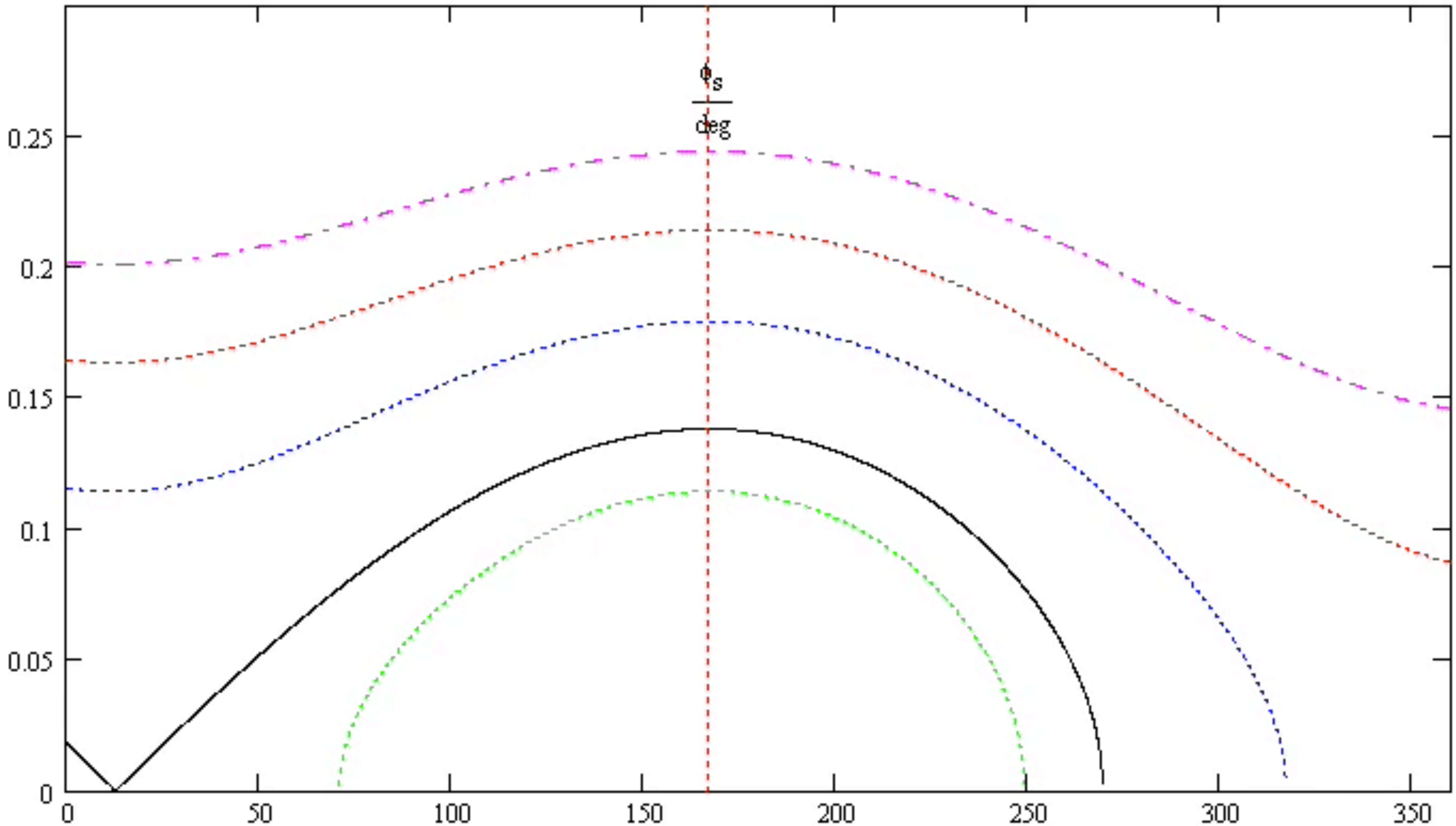

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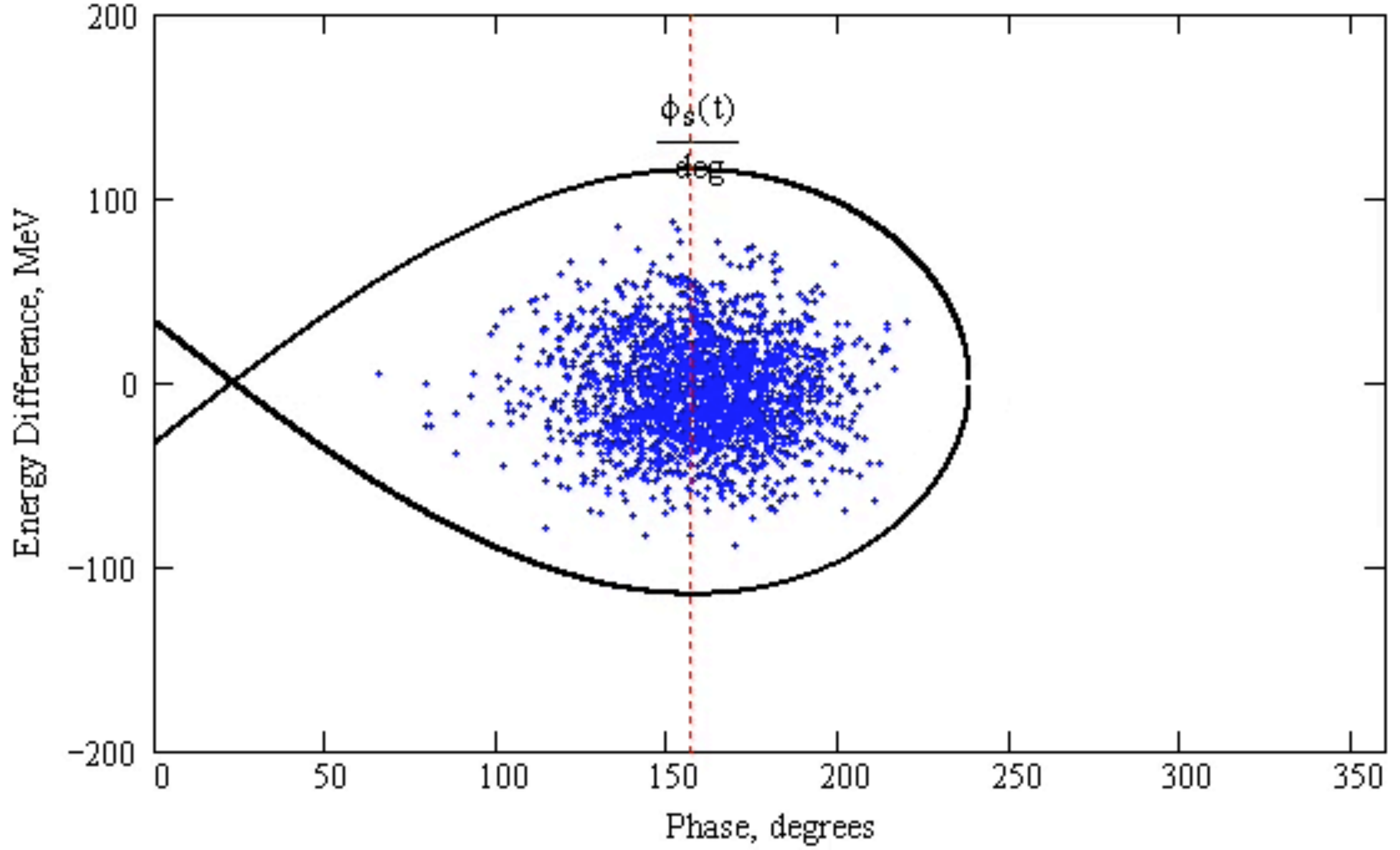
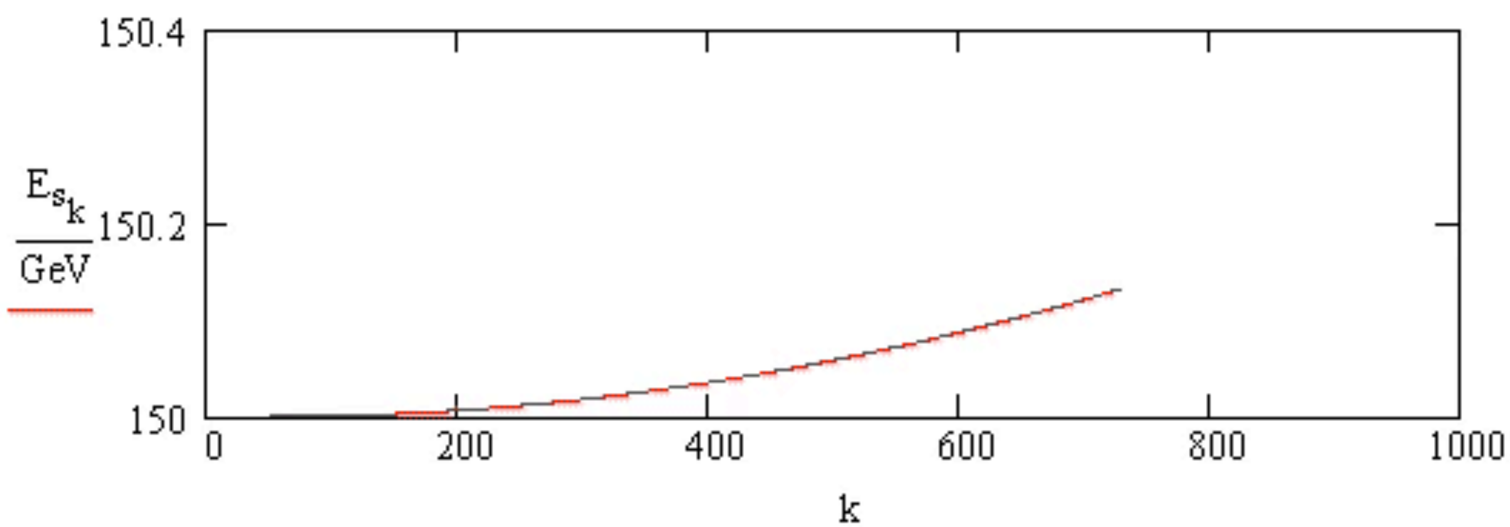
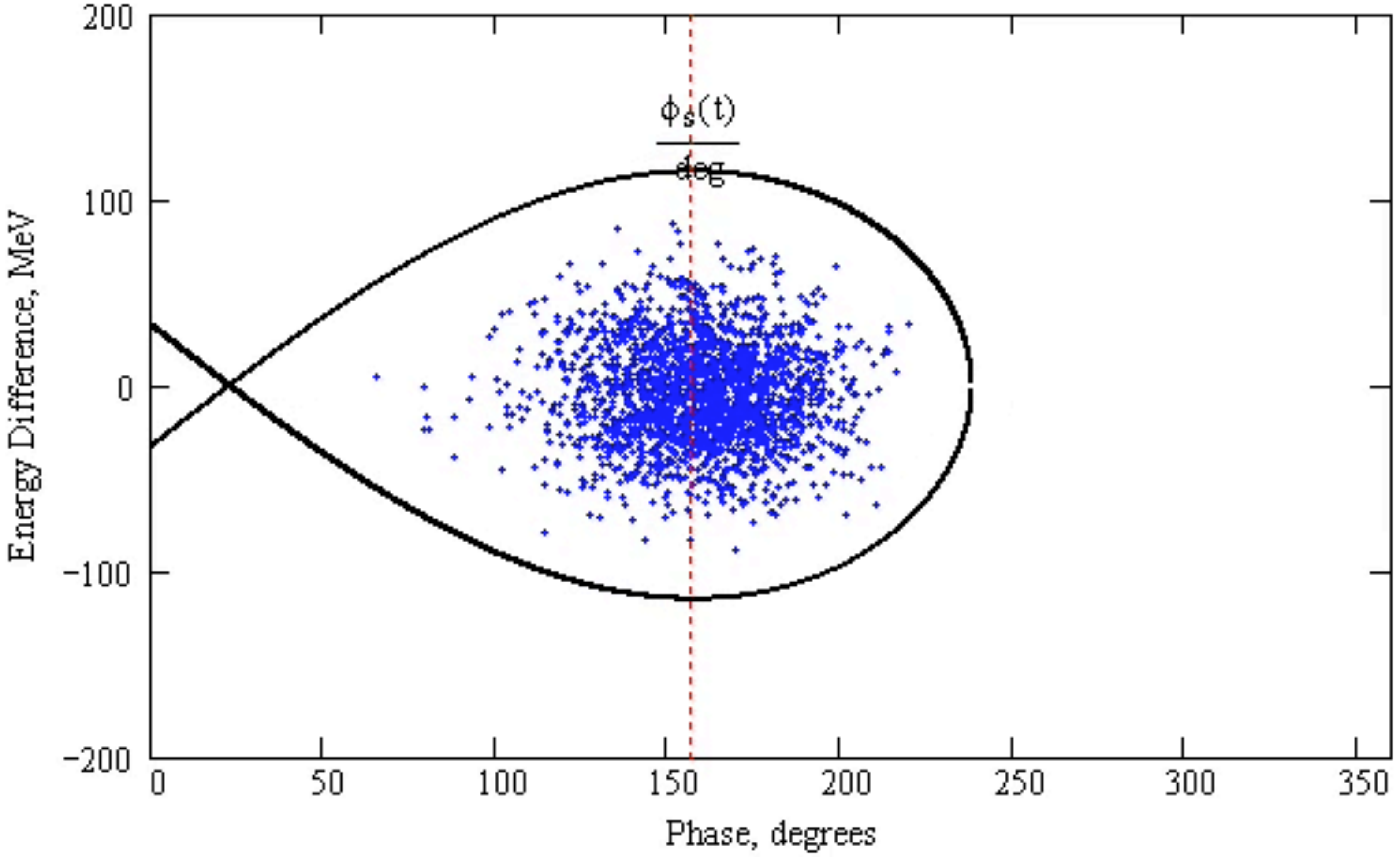
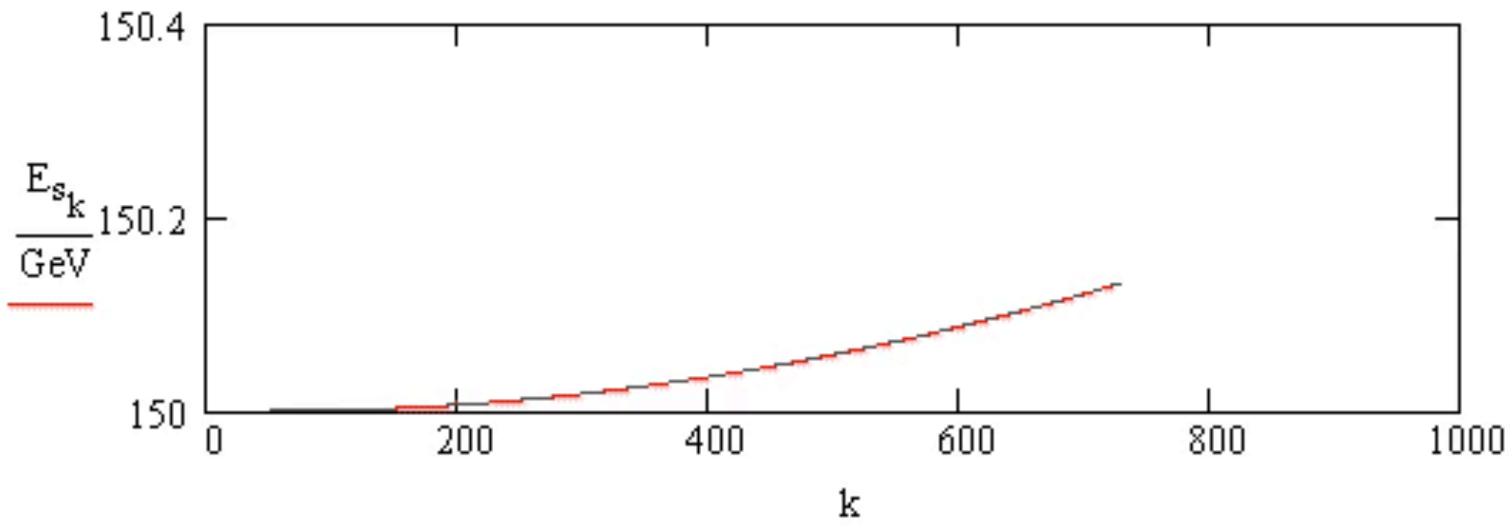


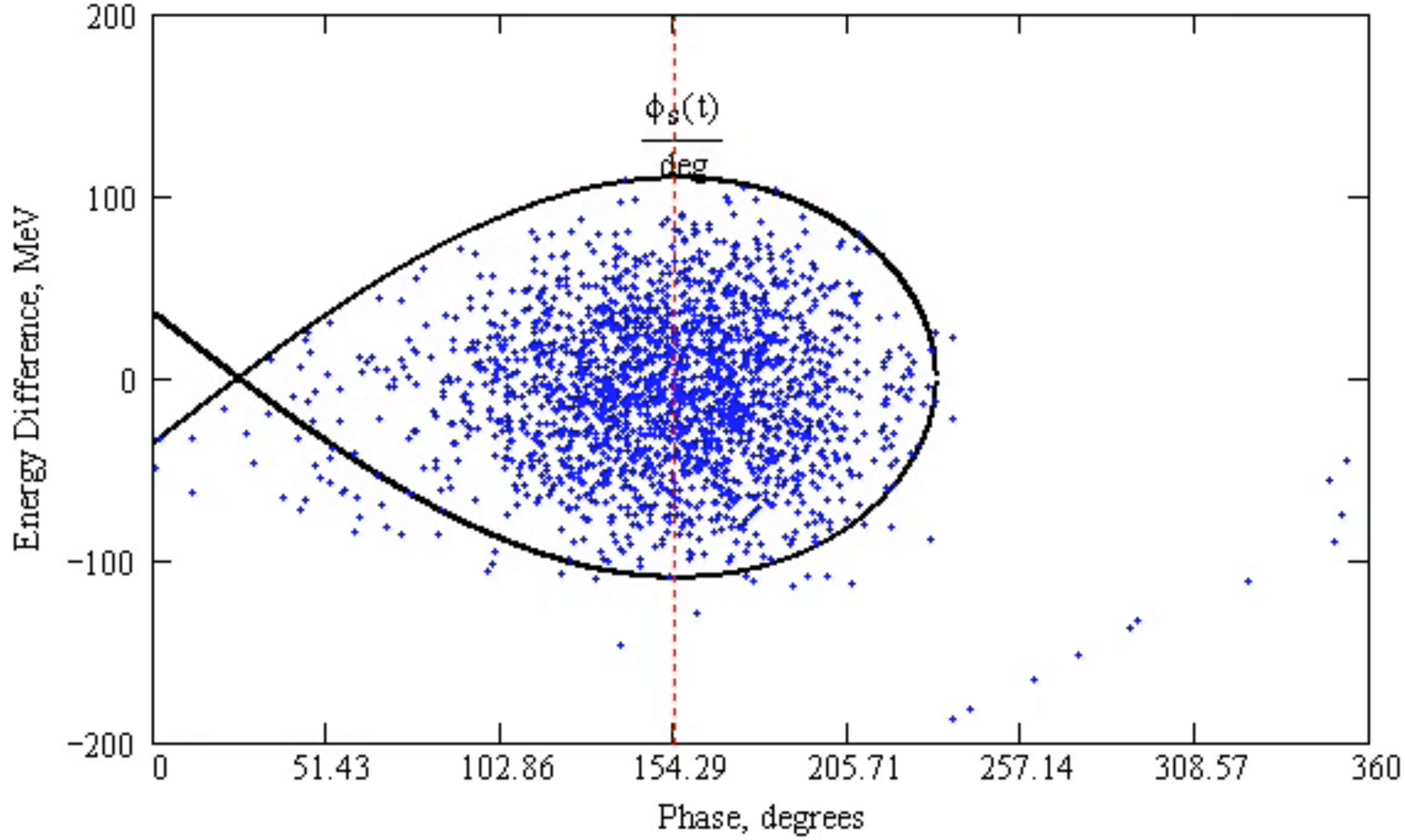




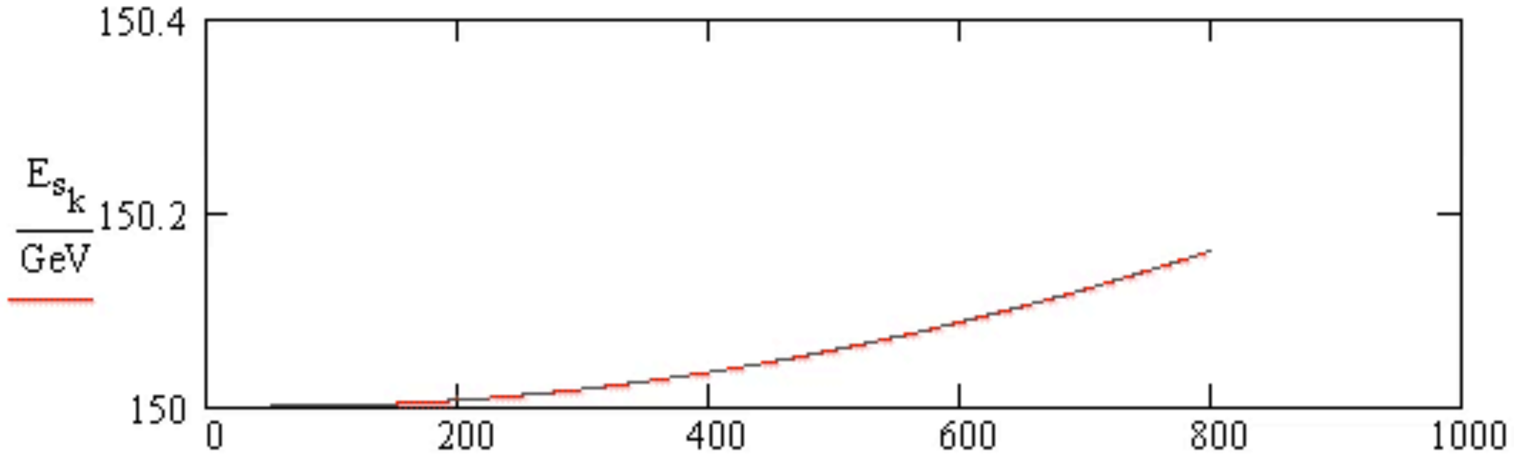
200
100
0
−100
−200
Energy Difference, MeV
0 50 100 150 200 250 300 350
Phase, degrees
$\frac{\phi_s(t)}{deg}$



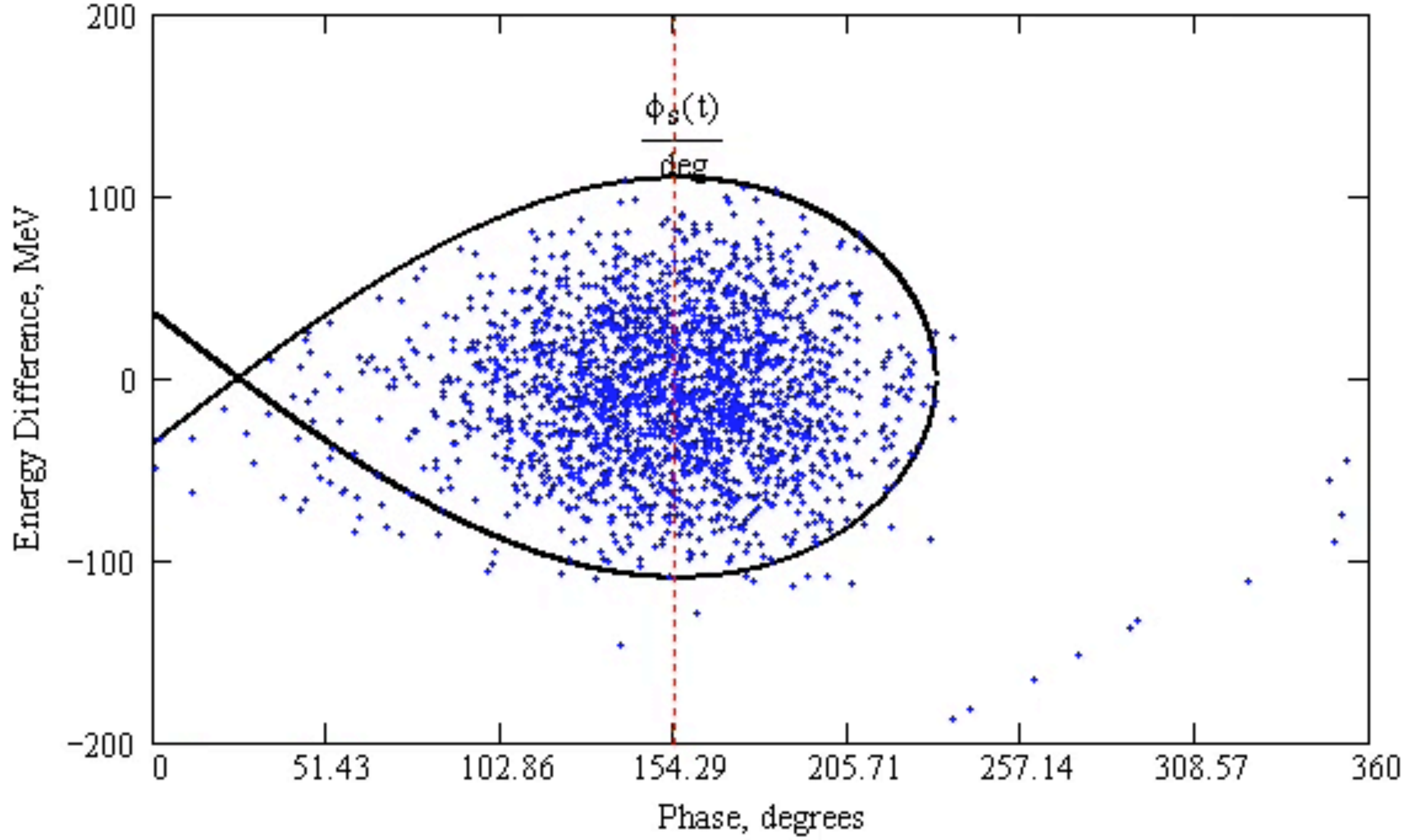
150.4
150.2
150
$\frac{E_{s_k}}{GeV}$
0 200 400 600 800 1000
k



200
100
0
−100
−200
0
51.43
102.86
154.29
205.71
257.14
308.57
360
Energy Difference, MeV
Phase, degrees
$\frac{\phi_s(t)}{deg}$



150.4
150.2
150
0
200
400
600
800
1000
$\frac{E_{s_k}}{GeV}$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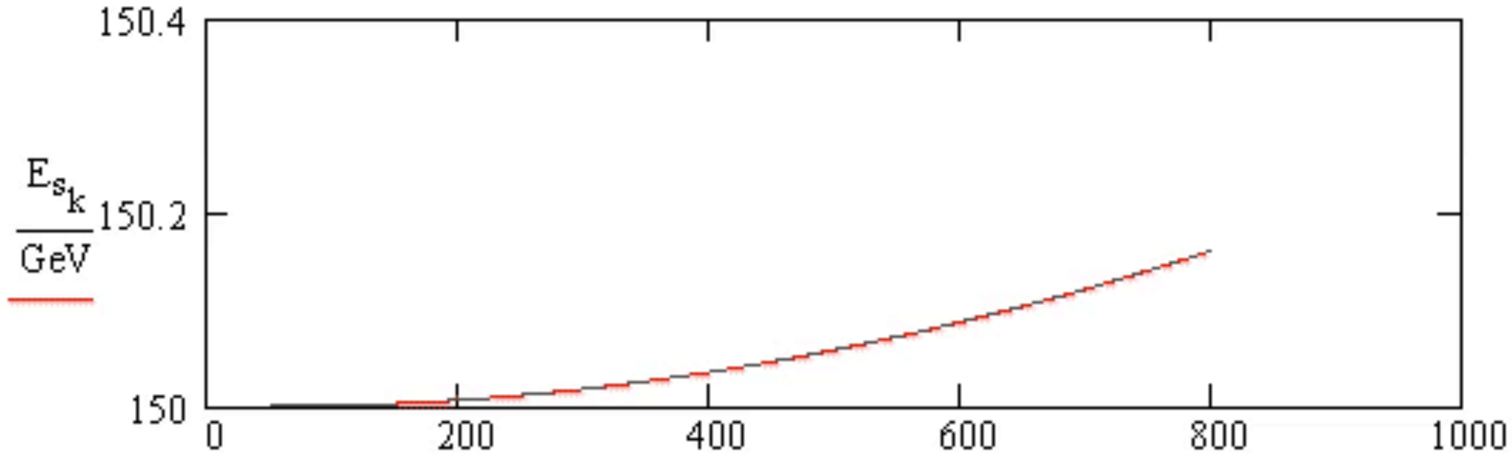
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150.4
150.2
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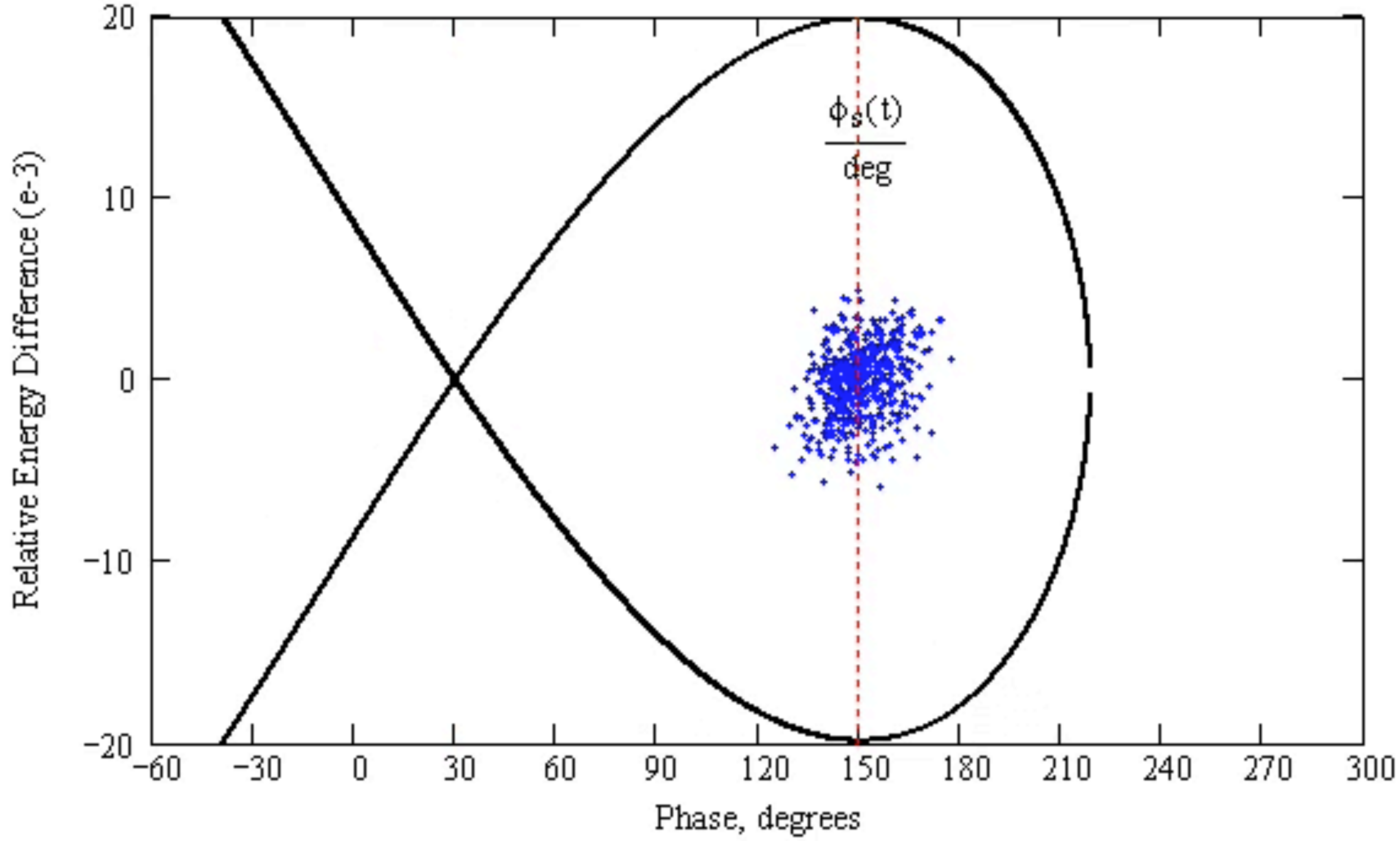
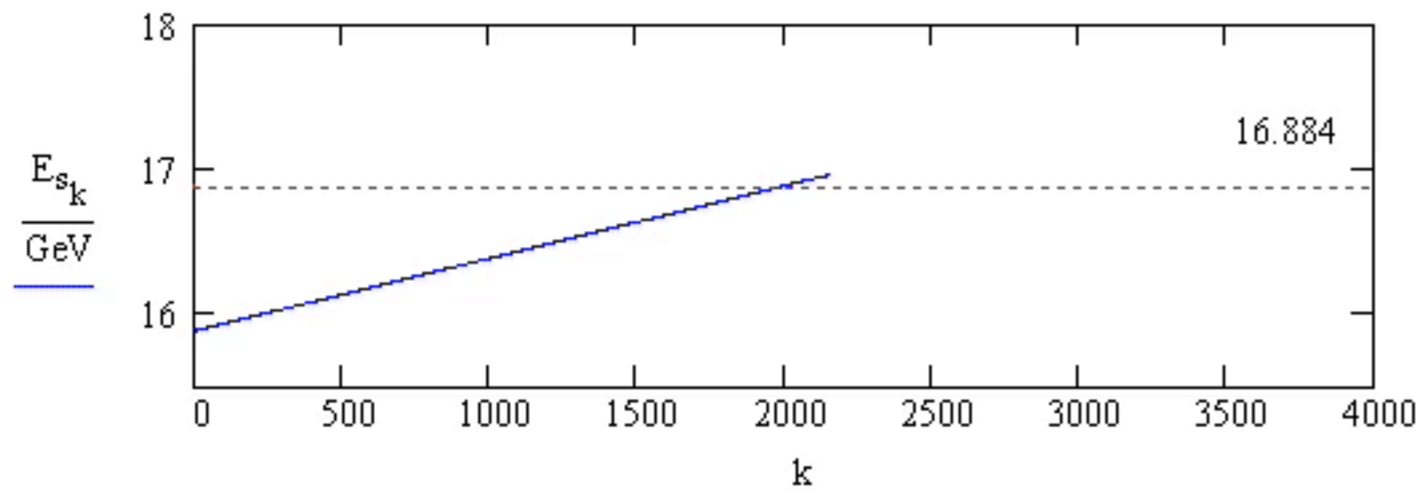
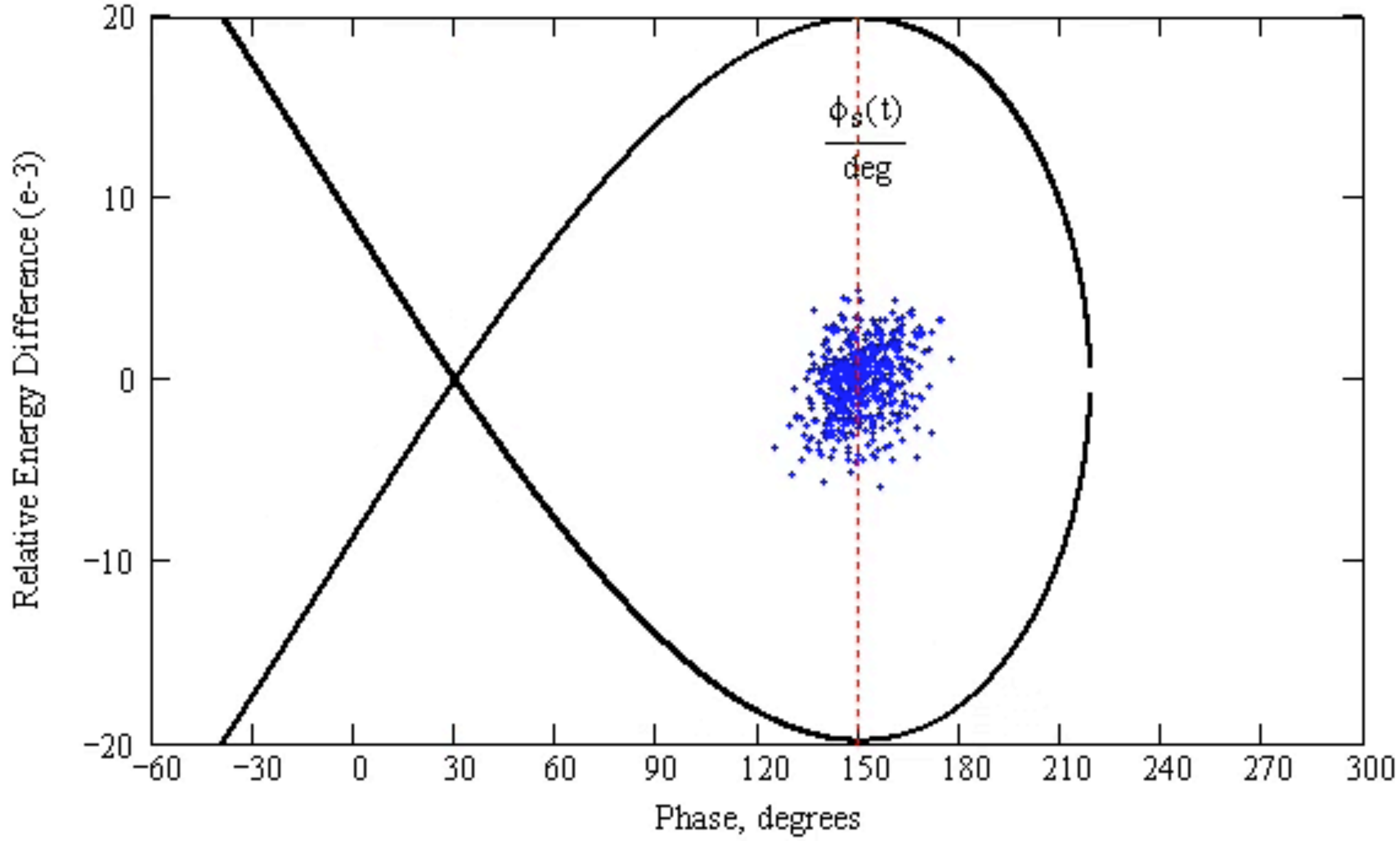
200
400
600
800
1000



$\sigma_{EonE_t} = 1.958 \times 10^{-3}$ $t = 2.161 \times 10^{3}$





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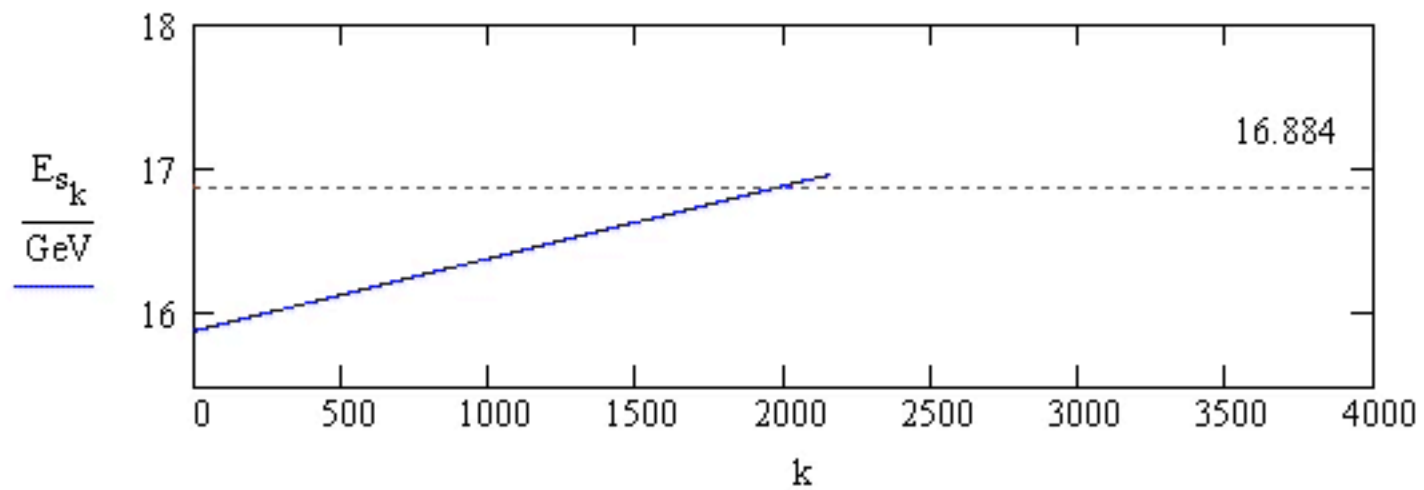
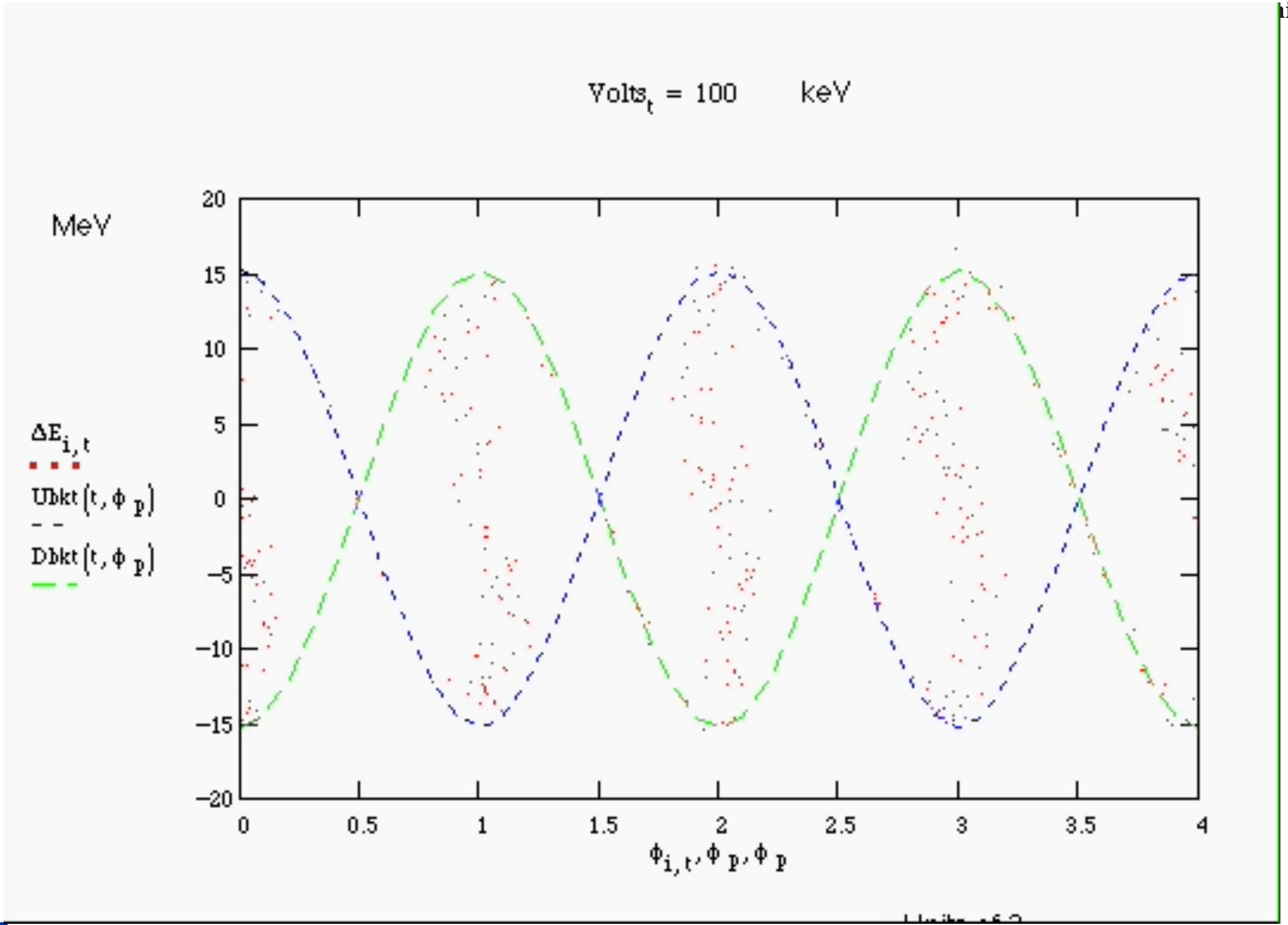
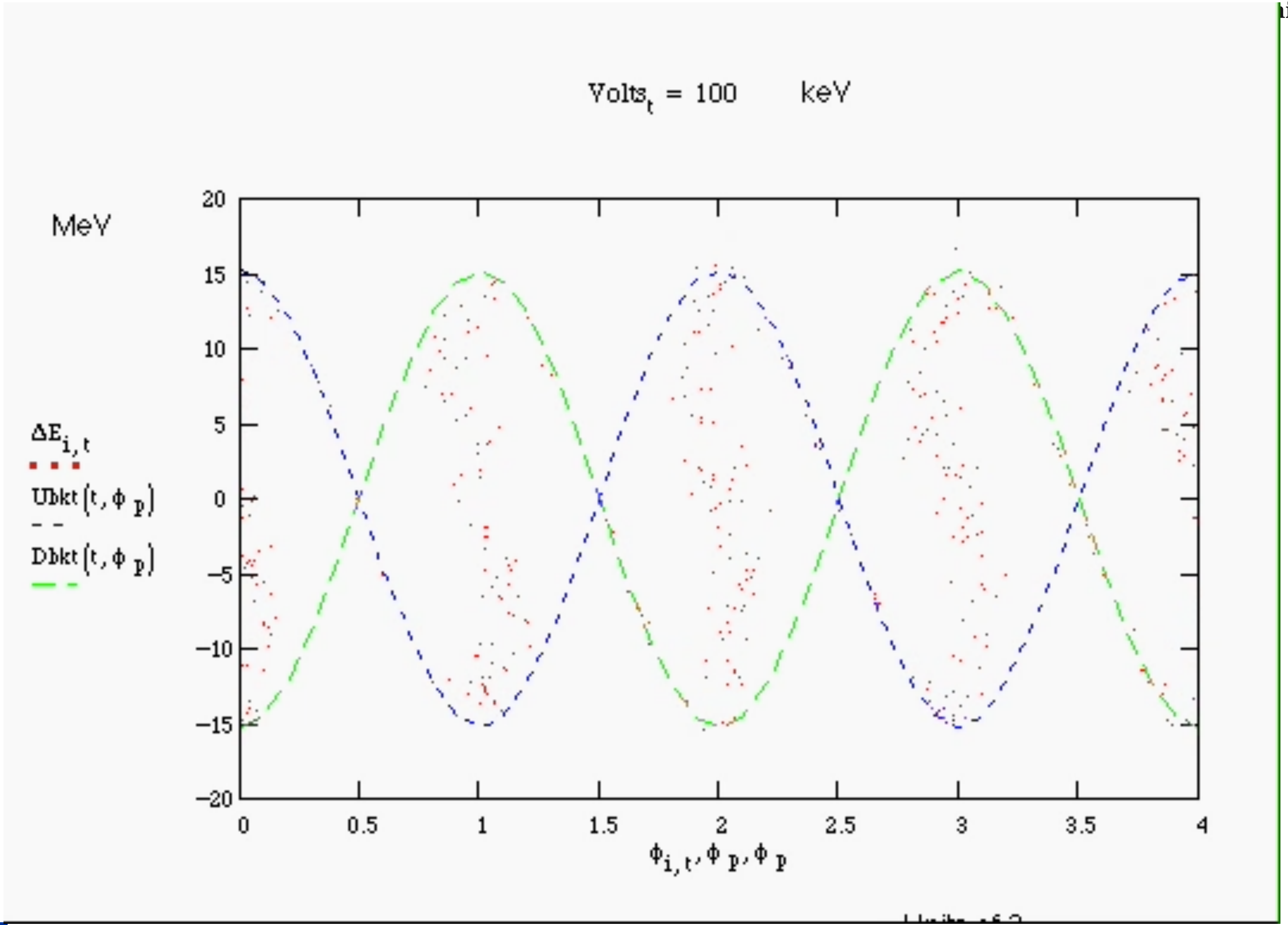



Volts$_t$ = 100 keV
MeV
$\Delta E_{i,t}$
$Ubkt(t, \phi_p)$
$Dbkt(t, \phi_p)$
20
15
10
5
0
−5
−10
−15
−20
0
0.5
1
1.5
2
2.5
3
3.5
4
$\phi_{i,t}, \phi_p, \phi_p$



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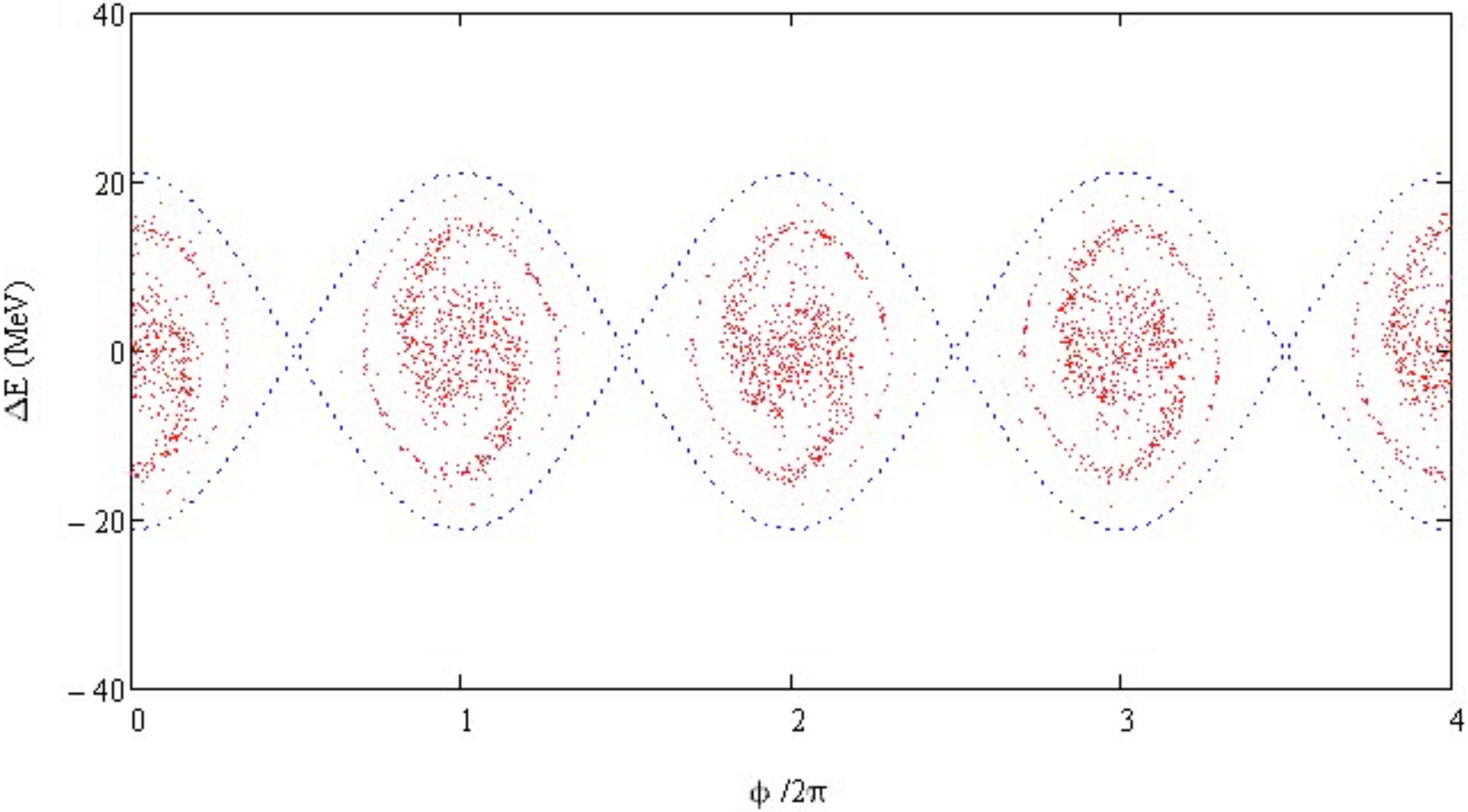
20
15
10
5
0
−5
−10
−15
−20
0
0.5
1
1.5
2
2.5
3
3.5
4
$\phi_{i,t}, \phi_p, \phi_p$

$eV(n) = 193.334\,keV$



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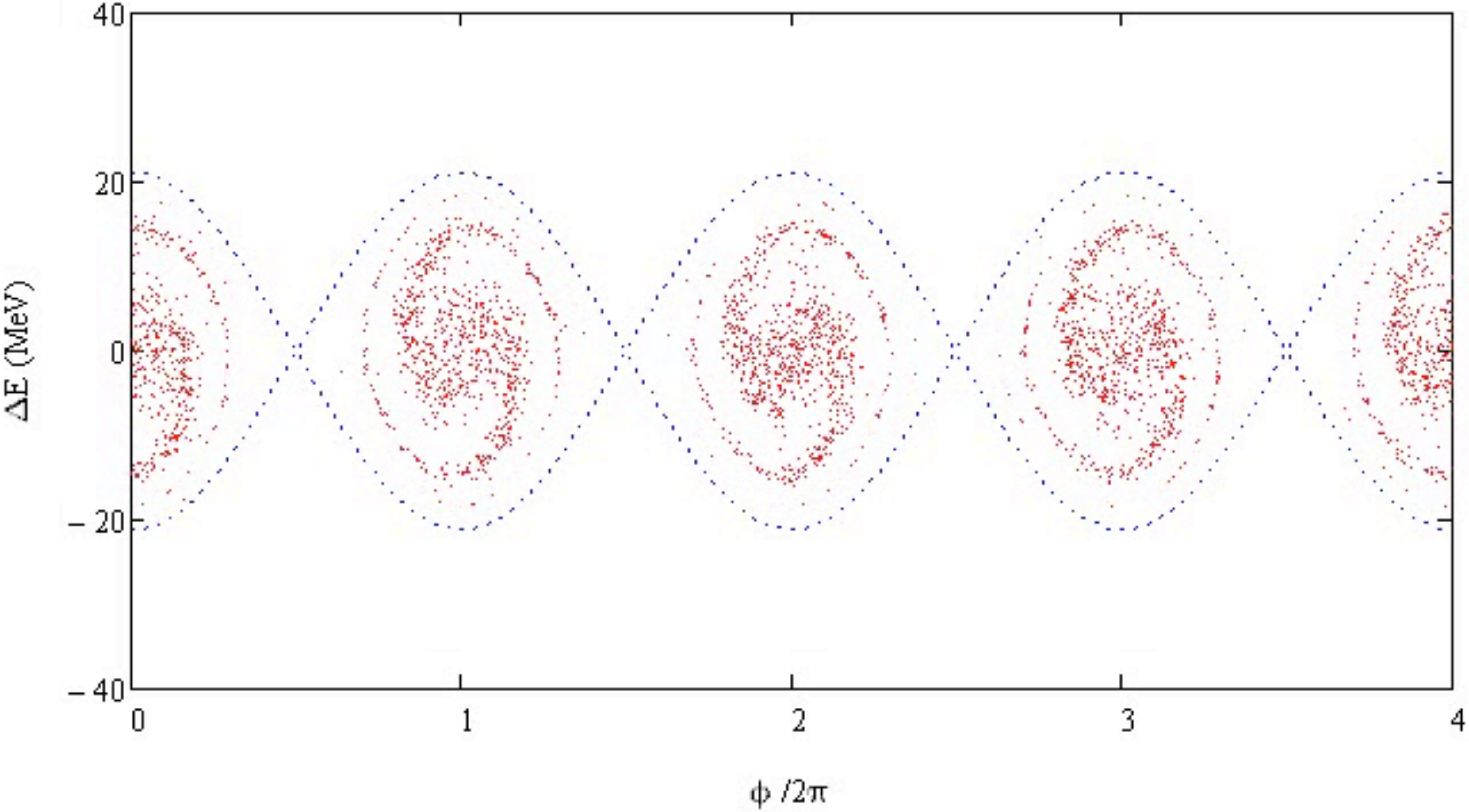

Discrete vs. Continuous Motion…

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- Since longitudinal motion is "slow", can usually treat time as differential variable
- However, acceleration happens at a "point" (or limited number of points) in the synchrotron; more accurate to treat as a "map":

$$\begin{aligned} \Delta E_{n+1} &= \Delta E_n + eV(\sin \omega_{\rm rf} \Delta t_n - \sin \phi_s) \\ \Delta t_{n+1} &= \Delta t_n + k\, \Delta E_{n+1} \end{aligned}$$

- Essentially the "Standard Map" (when $\phi_s = 0$)
  - (or Chirikov-Taylor map, or Chirikov standard map)

$$\begin{aligned} p_{n+1} &= p_n - K \sin \theta_n \\ \theta_{n+1} &= \theta_n + p_{n+1} \end{aligned}$$

Phase Space of the Standard Map

Northern Illinois University

- A Limit of Stability? *we know how to analyze this …*

$$
\begin{aligned}
p_{n+1} &= p_n - K \sin\theta_n \\
\theta_{n+1} &= \theta_n + p_{n+1}
\end{aligned}
$$

p

Each view uses the same initial conditions for 27 particles

Typical synchrotrons:

K ~ 0.0001 - 0.1

we had, for small synchrotron oscillations:

$$
\begin{aligned}
\Delta\phi_{n+1} &= \Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= QeV\cos\phi_s\Delta\phi_n + \left(1 + \frac{2\pi h\eta}{\beta^2 E}QeV\cos\phi_s\right)\Delta E_n
\end{aligned}
$$

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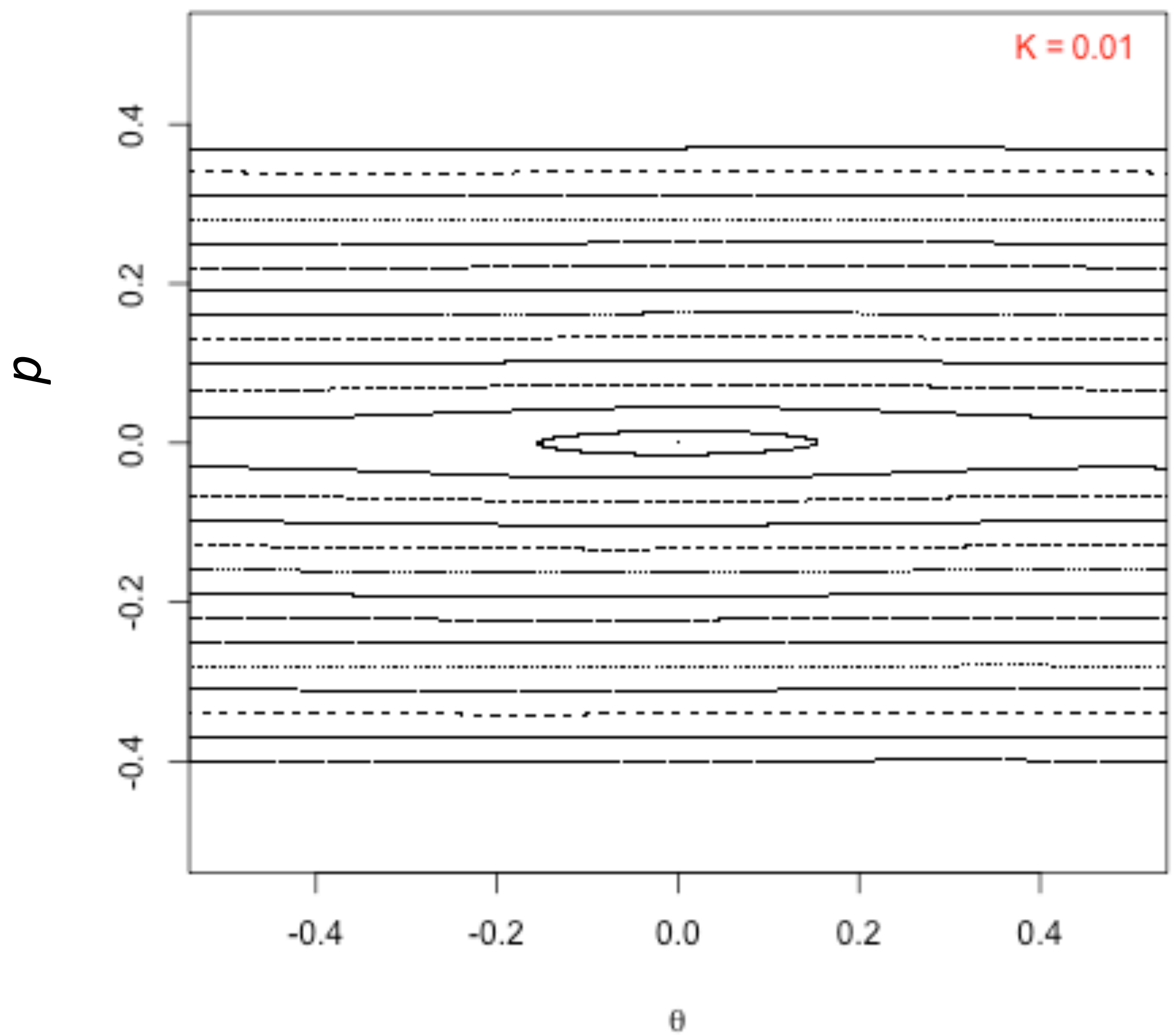

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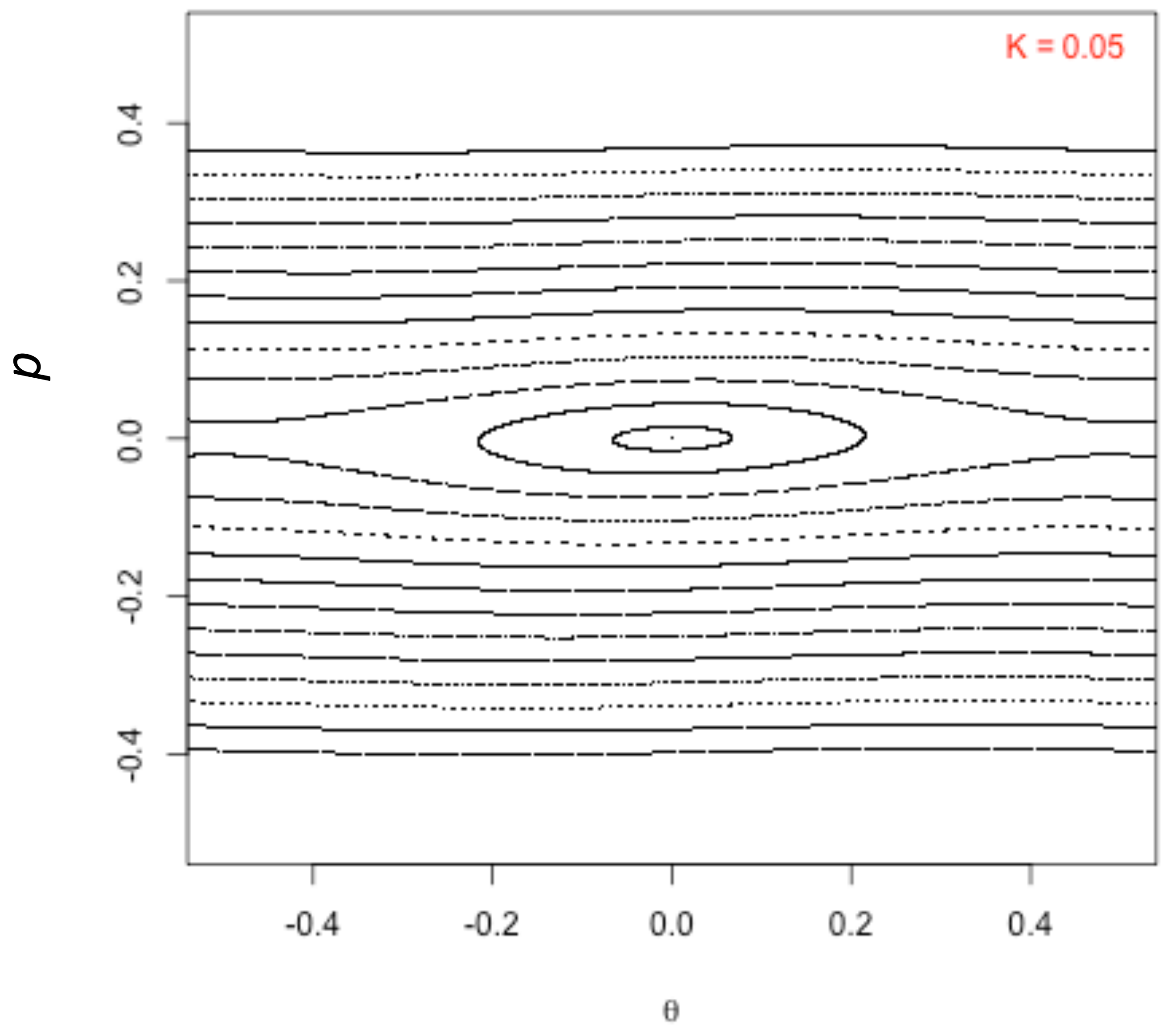

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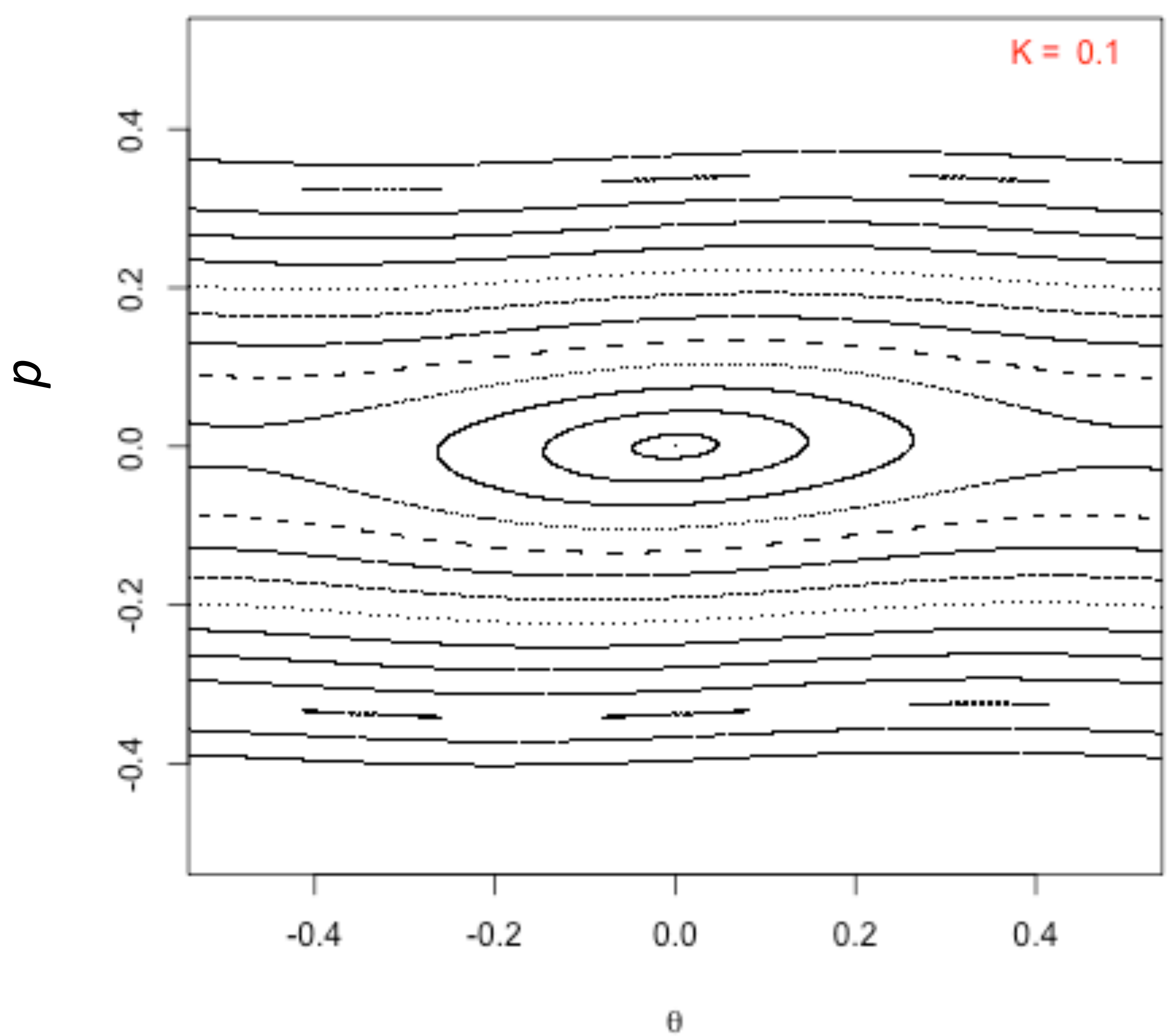

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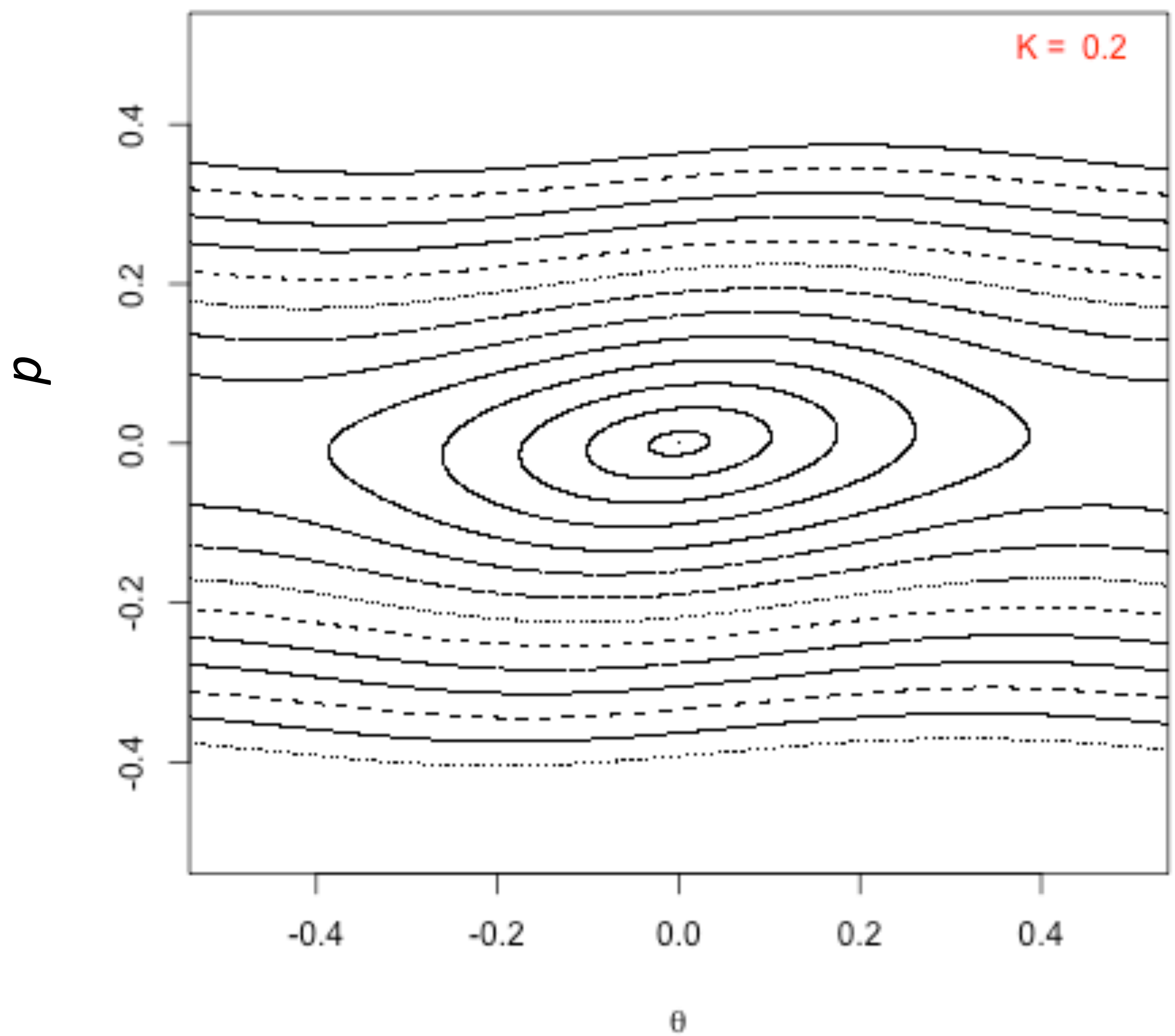

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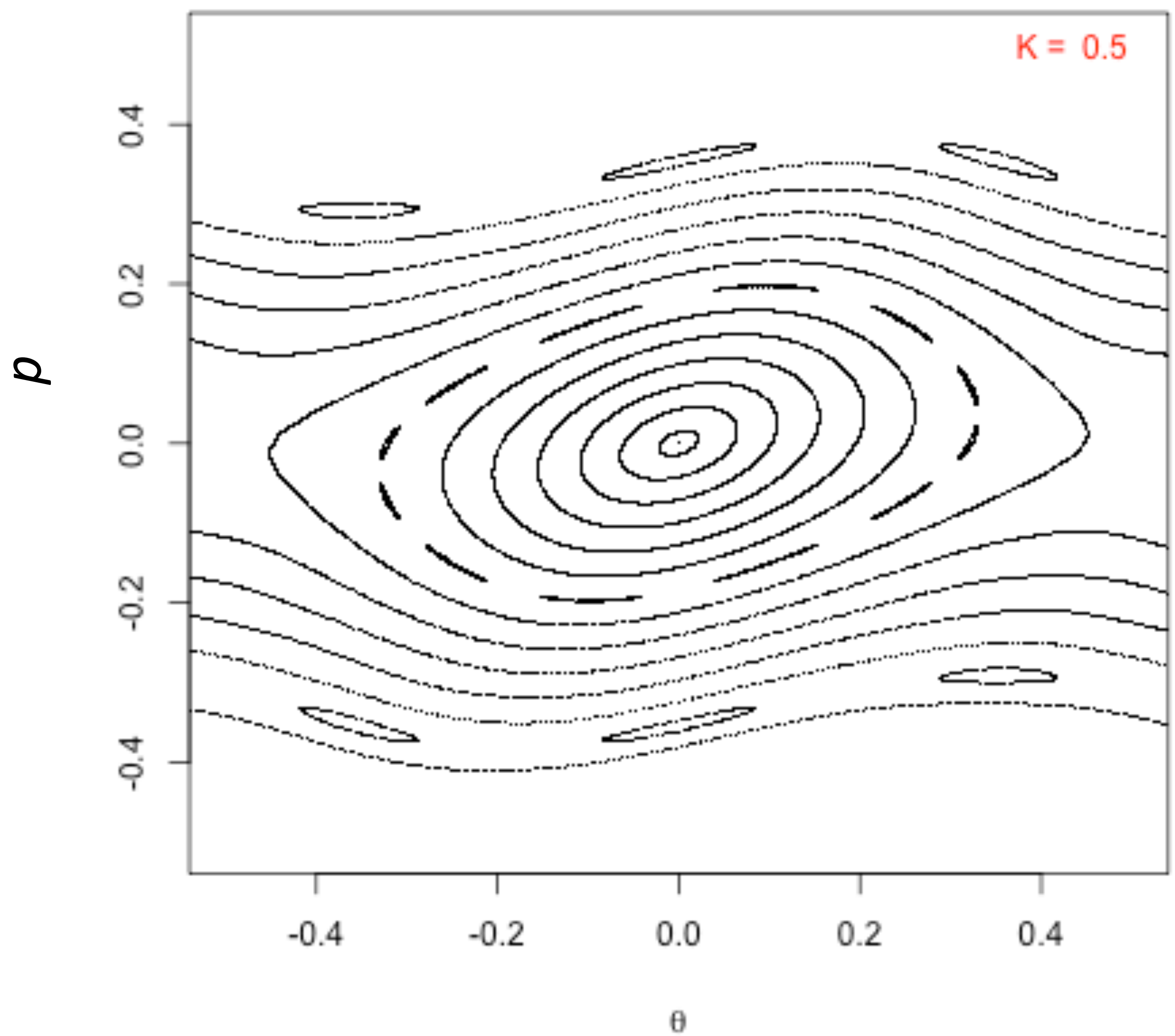

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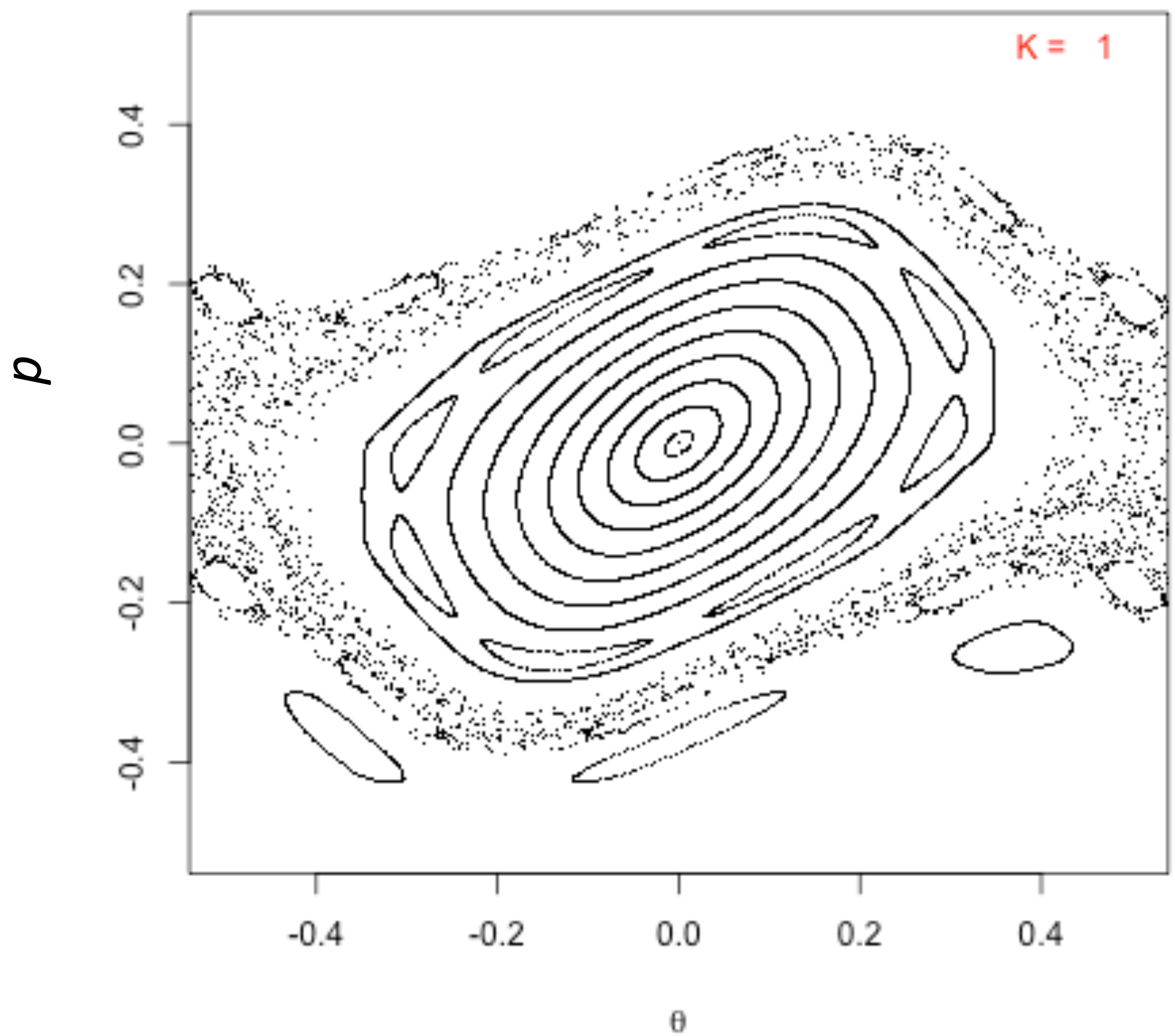

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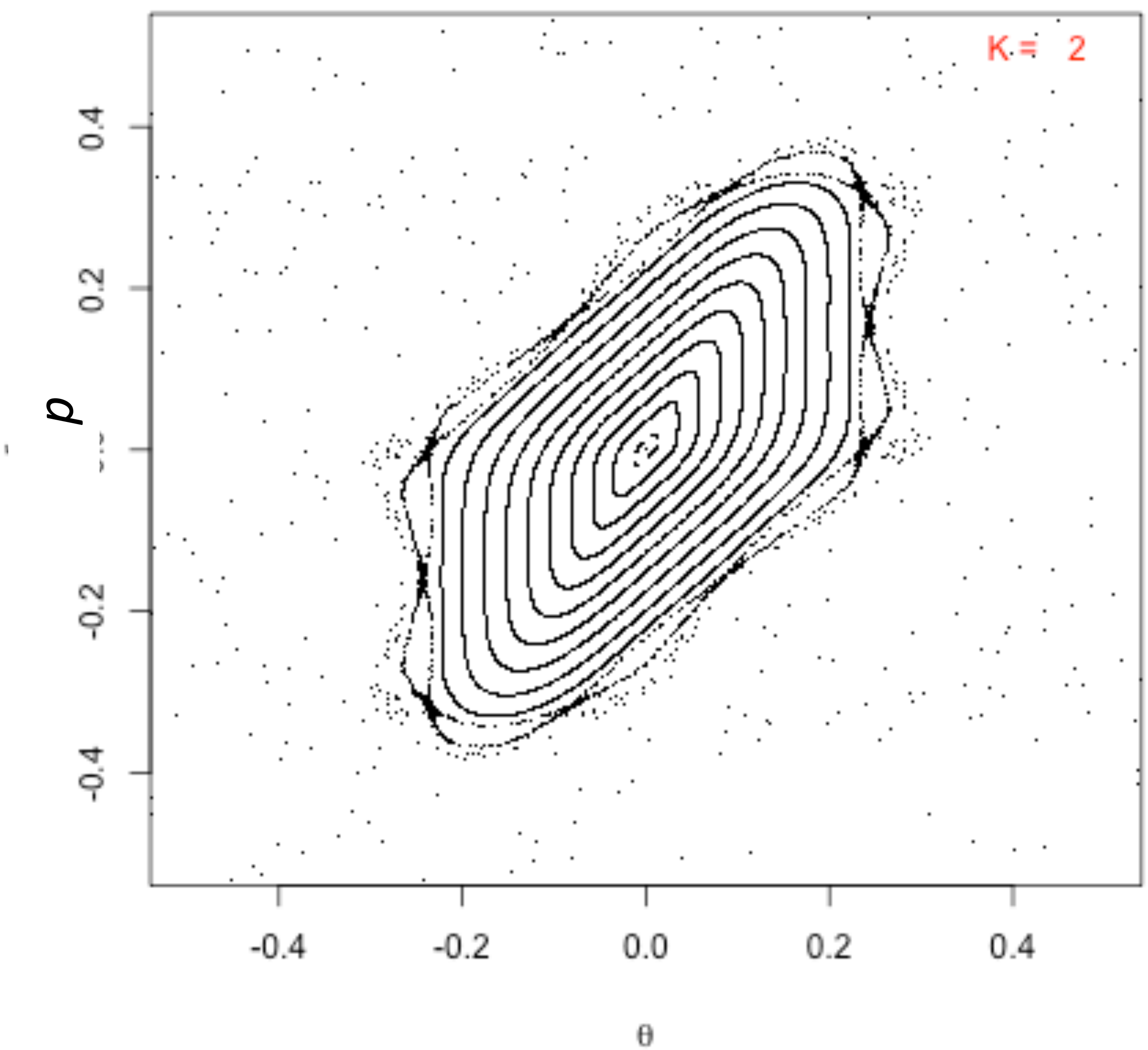
\begin{aligned}
\Delta\phi_{n+1} &= \Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= QeV\cos\phi_s\Delta\phi_n + \left(1 + \frac{2\pi h\eta}{\beta^2 E}QeV\cos\phi_s\right)\Delta E_n
\end{aligned}
$$

Phase Space of the Standard Map

Northern Illinois University

- A Limit of Stability? *we know how to analyze this …*



$$
\begin{aligned}
p_{n+1} &= p_n - K\sin\theta_n \\
\theta_{n+1} &= \theta_n + p_{n+1}
\end{aligned}
$$

Each view uses the same initial conditions for 27 particles

Typical synchrotrons:

K ~ 0.0001 - 0.1

we had, for small synchrotron oscillations:

$$
\begin{aligned}
\Delta\phi_{n+1} &= \Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= QeV\cos\phi_s\Delta\phi_n + \left(1 + \frac{2\pi h\eta}{\beta^2 E}QeV\cos\phi_s\right)\Delta E_n
\end{aligned}
$$

Let's analyze this….