Intensity Dependent Effects

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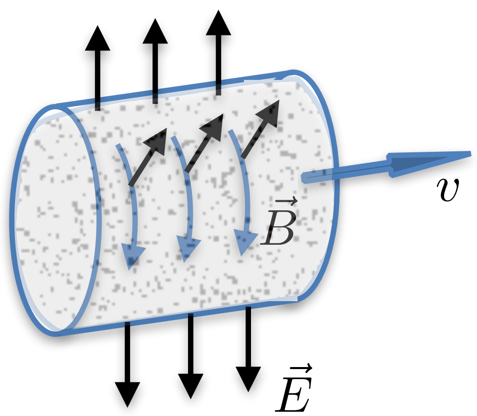
- The Space Charge Force
  - uniform, round beam of infinite extent
  - Gaussian (cross section) beam of infinite extent
  - tune spread due to space charge
  - The Beam-Beam force and tune spread in a collider
    - head-on collisions
    - long-range "collisions"
  - longitudinal fluctuations also cause longitudinal fields
    - can lead to various instabilities (e.g, "negative mass")

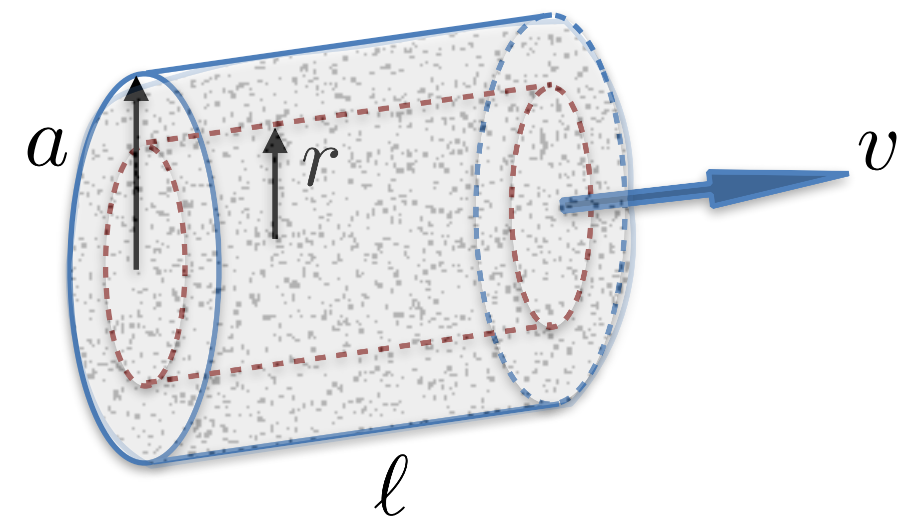
- Wake fields and impedance
  - Beam break-up in a long linac

Space Charge Effects at Very Low Energy

- Suppose have beam that is ***uniform*** in the direction of motion and transverse as well, out to a radius a

N' = no. particles/m along direction of motion





Gauss:

$$2\pi r E \ell = \frac{QeN'\ell}{\epsilon_0}\,(r^2/a^2) \quad \rightarrow \quad E = \frac{QeN'}{2\pi\epsilon_0\, a^2}\, r$$

Ampere:

$$2\pi r B = \mu_0\, QeN'v\,(r^2/a^2) \quad \rightarrow \quad B = \frac{\mu_0 QeN'v}{2\pi\, a^2}\, r \quad \rightarrow \quad \frac{QeN'v}{2\pi\epsilon_0 c^2 a^2}\, r$$

and so, …

$$\vec{F} = Qe[\vec{E} + \vec{v}\times\vec{B}] \quad \rightarrow \quad F = \frac{Q^2e^2N'}{2\pi\epsilon_0 a^2}\,(1 - v^2/c^2)\, r \quad \rightarrow \quad = \frac{Q^2e^2N'}{2\pi\epsilon_0 a^2\gamma^2}\, r$$

Space Charge Effects at Very Low Energy

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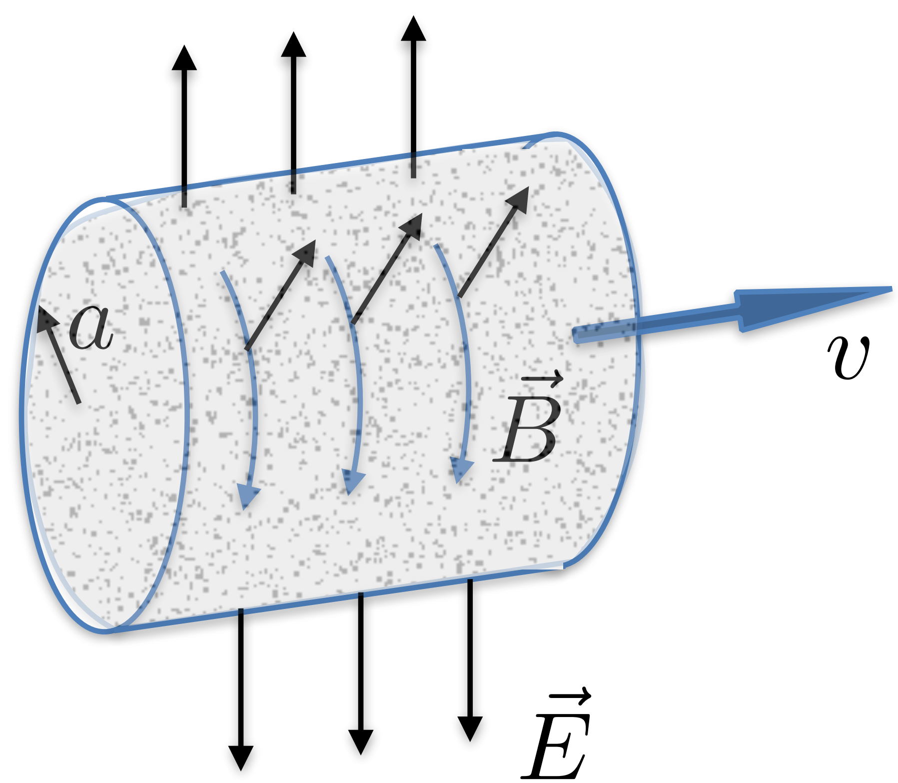
- Force ~ square of the charge

$$F = \frac{Q^2 e^2 N'}{2\pi\epsilon_0 a^2 \gamma^2}\, r \qquad (r < a)$$

- increases with longitudinal beam density
- is a *defocusing* effect
  - alters the effective optics



$$x'' + \left[K(s) - \frac{N' e^2 Q^2 c^2}{2\pi\epsilon_0 a^2 m v^2 \gamma^3}\right] x = 0$$

- effects decrease at higher energies
  - (i.e., larger values of γ)

Space Charge Force for Gaussian Beam

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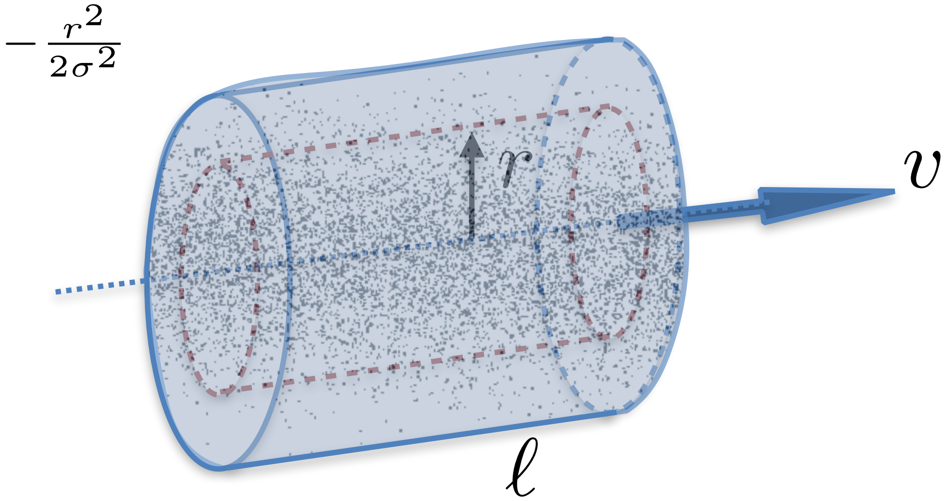
Rather than a uniform distribution, assume bi-Gaussian:

$$\frac{1}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}}$$



Gauss:

$$2\pi r E\ell = \frac{QeN'\ell}{2\pi\sigma^2\epsilon_0}\int_0^{2\pi}\int_0^{r} e^{-\frac{r^2}{2\sigma^2}}\; r\,dr\,d\theta$$

$$= \frac{QeN'\ell}{\epsilon_0}\left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)$$

$$F_E = \frac{Q^2e^2N'}{2\pi\epsilon_0}\;\frac{1 - e^{-\frac{r^2}{2\sigma^2}}}{r}$$

$$F_B = -\frac{Q^2e^2N'}{2\pi\epsilon_0}\;\frac{v^2}{c^2}\;\frac{1 - e^{-\frac{r^2}{2\sigma^2}}}{r}$$ (Ampere)

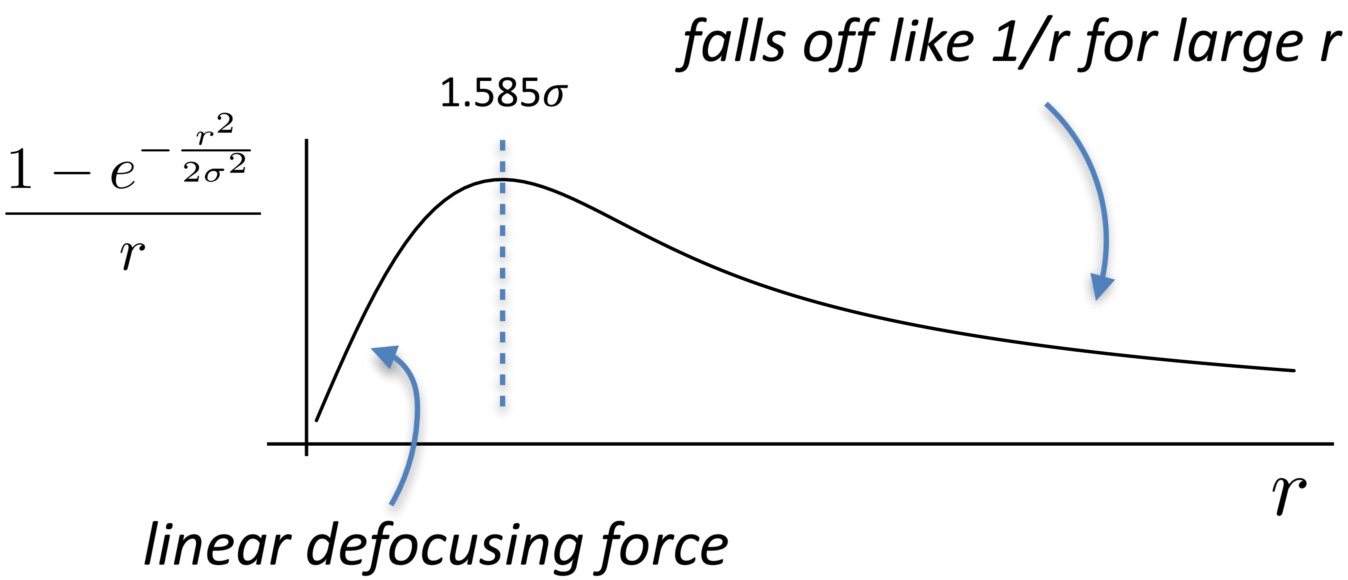
and so, …

$$F_{tot} = \frac{Q^2e^2N'}{2\pi\epsilon_0\gamma^2}\;\frac{1 - e^{-\frac{r^2}{2\sigma^2}}}{r} \quad\longrightarrow\quad \frac{Q^2e^2N'}{4\pi\epsilon_0\gamma^2\sigma^2}\; r$$

for $r << \sigma$

Space Charge Force for Gaussian Beam

$$F_{tot} = \frac{Q^2 e^2 N'}{2\pi\epsilon_0\gamma^2} \frac{1 - e^{-\frac{r^2}{2\sigma^2}}}{r}$$



- For a large portion of the beam, there will be a linear defocusing force. If the local beam intensity is very large, the effects can be strong.
- Even if the intensity is modest, there can be an effect as the beam particles travel long distances together
  - lowering of the betatron tune in a ring, for example

Space Charge Tune Shift

- In a synchrotron, the space charge force is constantly defocusing the beam — leads to a decrease in the betatron tune in each plane
- Use tune shift formula to estimate the effect…

$$\Delta\nu = \frac{1}{4\pi}\oint \beta(s)K(s)ds$$

$$F \longrightarrow \frac{Q^2e^2N'}{4\pi\epsilon_0\gamma^2\sigma^2}\, r$$

Here, $K(s) = \frac{\frac{\partial B_y}{\partial x}}{B\rho} \Rightarrow \frac{\partial F/\partial r}{qv\cdot p/q} = \frac{\partial F/\partial r}{pv}$

$$\approx \frac{Q^2e^2N'}{4\pi\epsilon_0\gamma^2\sigma^2\cdot\gamma\beta^2mc^2} = \frac{Q^2e^2N'}{4\pi\epsilon_0 mc^2\gamma^3\beta^2\sigma^2(s)}$$

Space Charge Tune Shift

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$$\Delta\nu = \frac{1}{4\pi}\frac{Q^2 N' r_0}{\gamma^3\beta^2}\int_0^C \frac{\beta(s)}{\sigma^2(s)}ds$$

$$= \frac{Q^2 N' r_0}{4\gamma^2\beta}\int_0^C \underbrace{\frac{\beta(s)}{\pi\sigma^2(s)\cdot\beta\gamma}ds}_{=\frac{C}{\epsilon_N}}$$

$$K(s) = \frac{Q^2 e^2 N'}{4\pi\epsilon_0 mc^2\gamma^3\beta^2\sigma^2(s)}$$

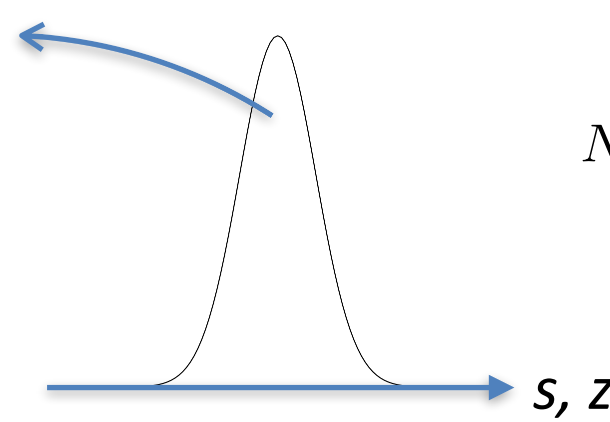
$$r_0 \equiv \frac{e^2}{4\pi\epsilon_0 mc^2}$$

"classical radius" of the particle

$$\boxed{\Delta\nu = \frac{Q^2 r_0 N' C}{4\epsilon_N\gamma^2\beta}}$$

Note: *N'* = no. particles / m

N = no. particles / bunch



$$N'_{max} = \frac{N}{\sqrt{2\pi}\sigma_z}$$

Space Charge Tune Shift

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$$\Delta\nu = \frac{Q^2 r_0 N' C}{4\epsilon_N \gamma^2 \beta}$$

- Estimate for a particular synchrotron:
  - Fermilab Booster Ring at injection
    - » r_0 = 1.53x10^{-18} m, $N'C$ = (10^{10}/m)2π(75 m)
    - » ϵ_N = 1π mm-mr, W=400 MeV (β=0.7, γ=1.4)
    - » Thus, we get $\Delta\nu \approx 0.4$!
  - in early 1990s, the injection energy was raised from 200 to 400 MeV and intensity increased
- Note, too: the effect scales with Q^2. Can be more detrimental for beams with high charge states (though they often have less intensity)

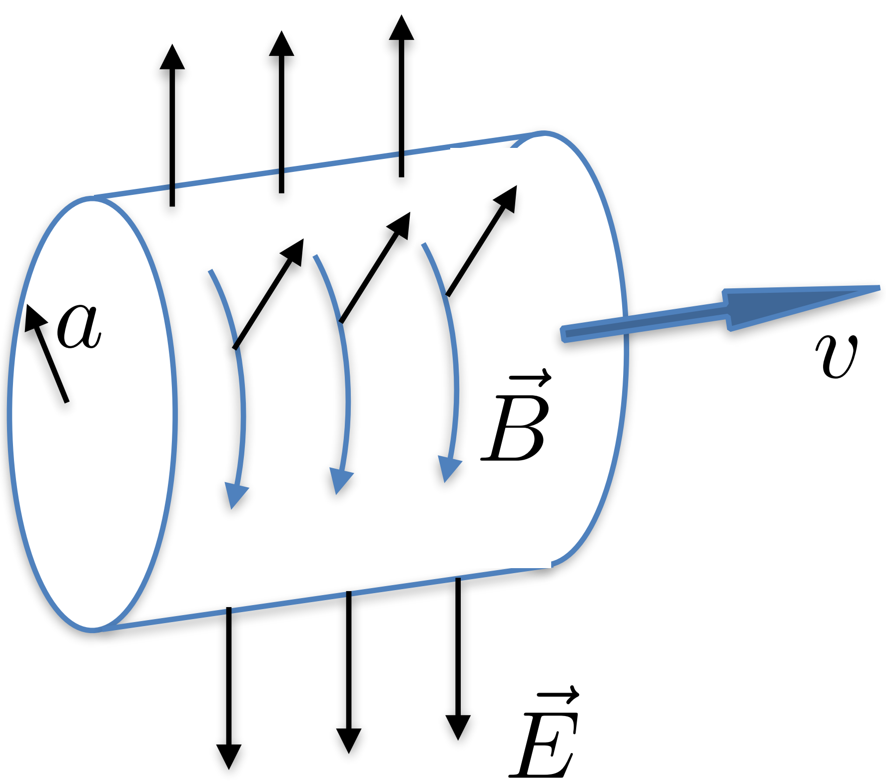
Space Charge Tune Shift

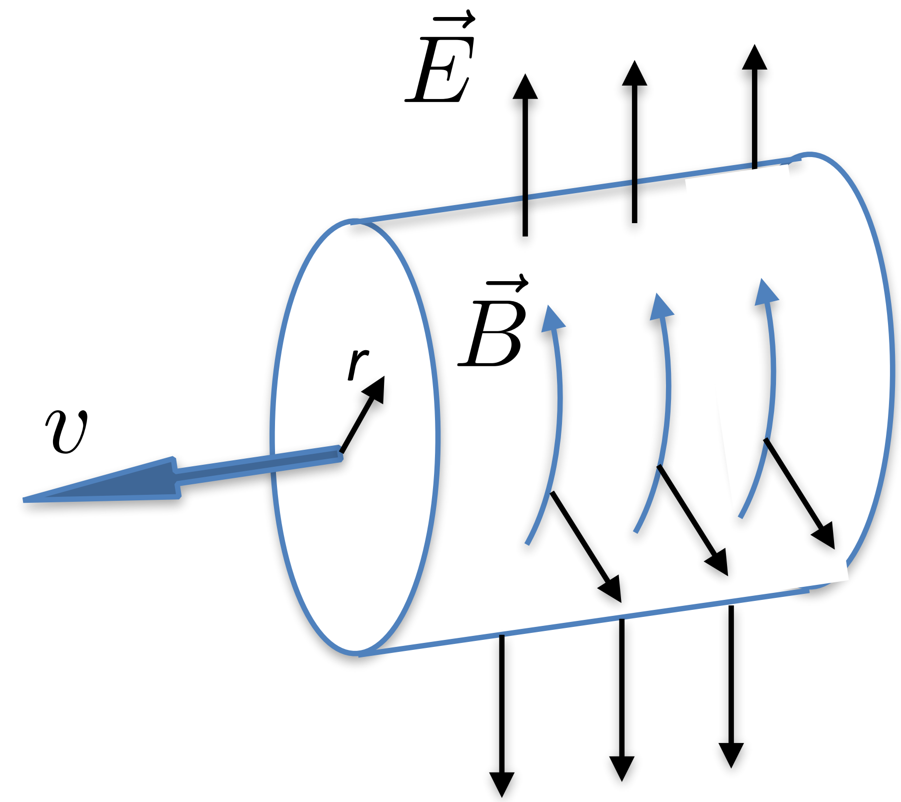
$$\Delta\nu = \frac{Q^2 r_0 N' C}{4\epsilon_N \gamma^2 \beta}$$

- We have estimated the shift in the betatron tune of the particles near the center of the distribution. Those at the far edges of the distribution will have little or no change in their tunes. Thus, the space charge force will actually generate a **spread** in tune among the particles. This spread in tune will need to fit in between the various resonance lines in the *x-y* tune space in order to avoid resonances and ultimately beam loss

    - increase injection energy to reduce the tune spread, hopefully be able to transport more beam

Beam-Beam Interactions





$$E = \frac{QeN'}{2\pi\epsilon_0\ a^2}\ r$$

$$B = \frac{\mu_0 QeN'v}{2\pi\ a^2}\ r \quad \rightarrow \quad \frac{QeN'v}{2\pi\epsilon_0 c^2 a^2}\ r$$

Here, *v* of the "test" particle is opposite that of the other beam

Beam-Beam Tune Shift in a Collider

- Behaves similarly to the space charge tune shift just derived previously, except…
  - only occurs over the interaction length of the two bunches passing through each other
    - » interaction length = ***1/2*** the bunch length
  - the forces from the *E* field and the *B* field add up, rather than subtract
    - » rather than $1/\gamma^2$, we get 2! (for $v \approx c$)
- Thus, the beam-beam force will not depend upon energy, and it only occurs during the interaction, not all along the circumference.

Beam-Beam Tune Shift in a Collider

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- Previous Space Charge Calculation:

$$\vec{F} = Qe[\vec{E} + \vec{v} \times \vec{B}] \;\rightarrow\; F = \frac{Q^2 e^2 N'}{2\pi\epsilon_0 a^2}\left(1 - v^2/c^2\right) r \;\rightarrow\; = \frac{Q^2 e^2 N'}{2\pi\epsilon_0 a^2 \gamma^2}\, r$$

- For the Beam-Beam Interaction

$$\vec{F} = Qe[\vec{E} + \vec{v} \times \vec{B}] \;\rightarrow\; F = \frac{Q^2 e^2 N'}{2\pi\epsilon_0 a^2}\left(1 + v^2/c^2\right) r \;\rightarrow\; = \frac{Q^2 e^2 N'}{\pi\epsilon_0 a^2}\, r$$

Then, for Gaussian beam:

$$\Delta\nu = \frac{Q^2 r_0}{2} \int_0^{\ell/2} \frac{N'\beta(s)}{\pi\sigma^2(s)\cdot\beta\gamma}\, ds$$

N = no. particles / bunch

$$\boxed{\Delta\nu_{bb} = \frac{Q^2 r_0 N}{4\epsilon_N}}$$

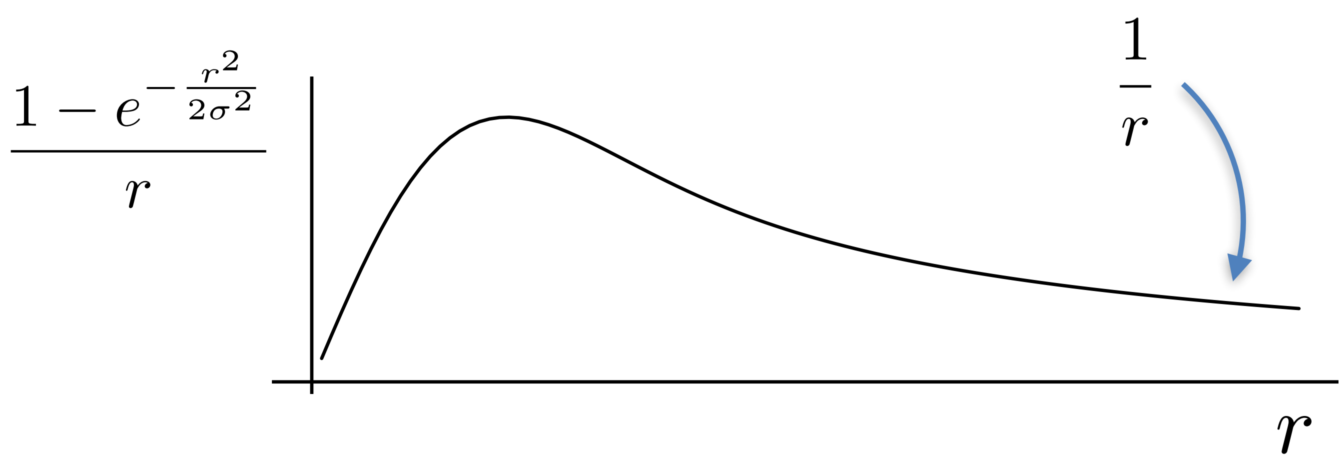
Typical range of values:
~0.01, proton colliders
~0.1, electron colliders

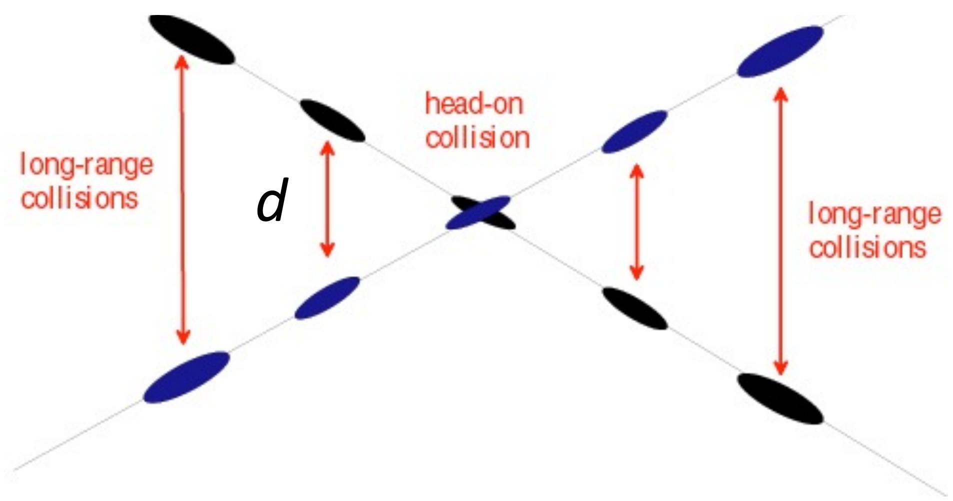
Note:

if the colliding particles have opposite signs, will be a focusing effect rather than defocusing

Long-Range Beam-Beam Interactions

- Long-range force: $r >> \sigma$





crossing angle, θ

$d \sim \theta L$

and want $d \sim n\sigma = n\sigma^* (\beta/\beta^*)^{1/2}$

but $(\beta/\beta^*)^{1/2} \sim L/\beta^*$

Thus, the crossing angle:

$\theta \sim n\ \sigma^*/\beta^*$

typically choose $n \sim$ 10-12

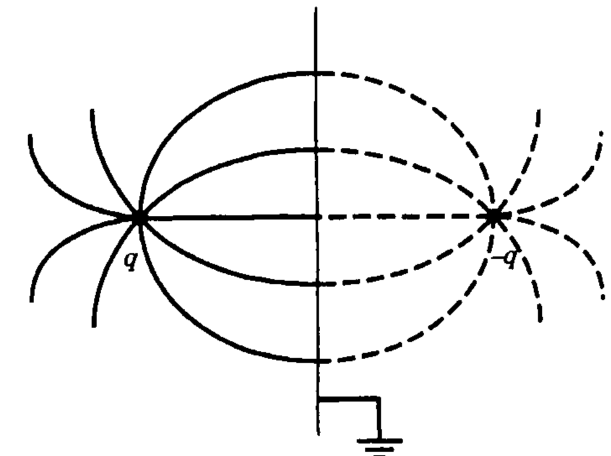
If have a central head-on tune spread of $\Delta\nu_{bb} = \frac{Q^2 r_0 N}{4\epsilon_N}$

then **each** long-range interaction will generate a tune shift of

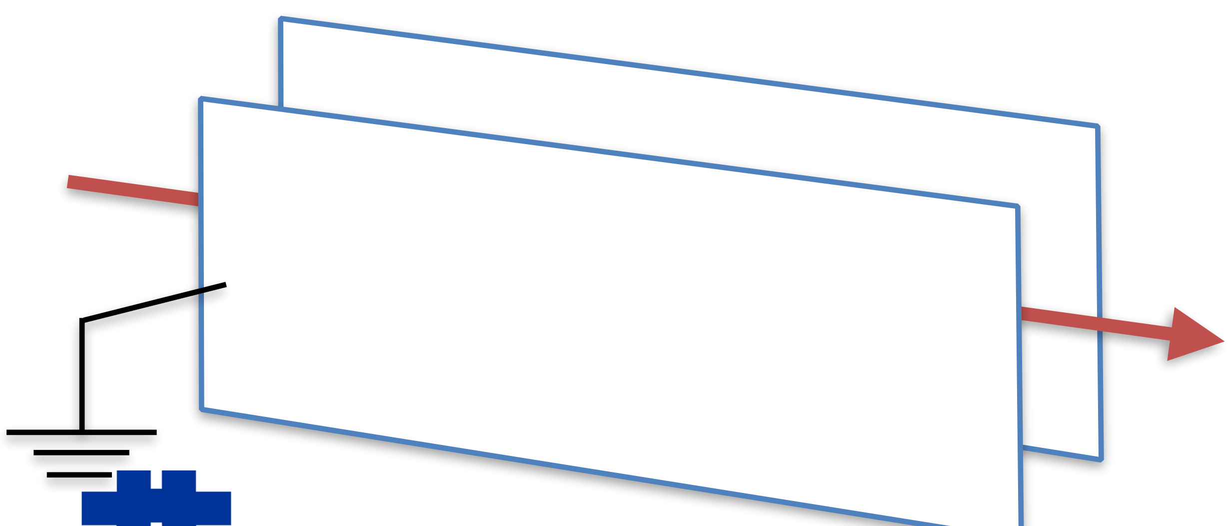
$$\Delta\nu_{LR} = \frac{2\Delta\nu_{bb}}{(d/\sigma)^2}$$

Image Charges and Currents

- When intense beams move through a pipe or chamber made of grounded conducting surfaces, image charges and their subsequent currents are induced in the walls which create forces that act back onto the beam.



- Imagine a line charge contained between two parallel conducting plates:



$\lambda = dq/dz$
$= QeN'$

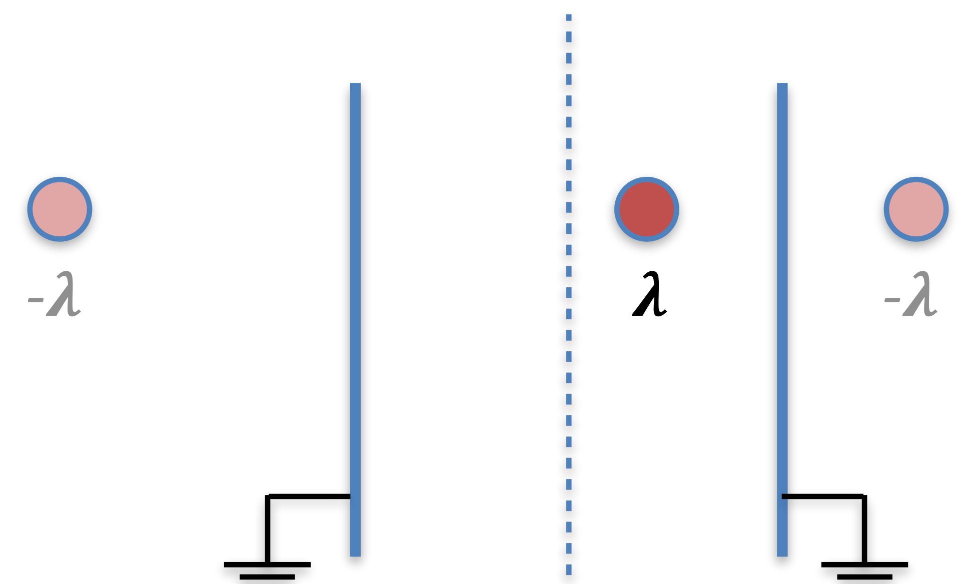
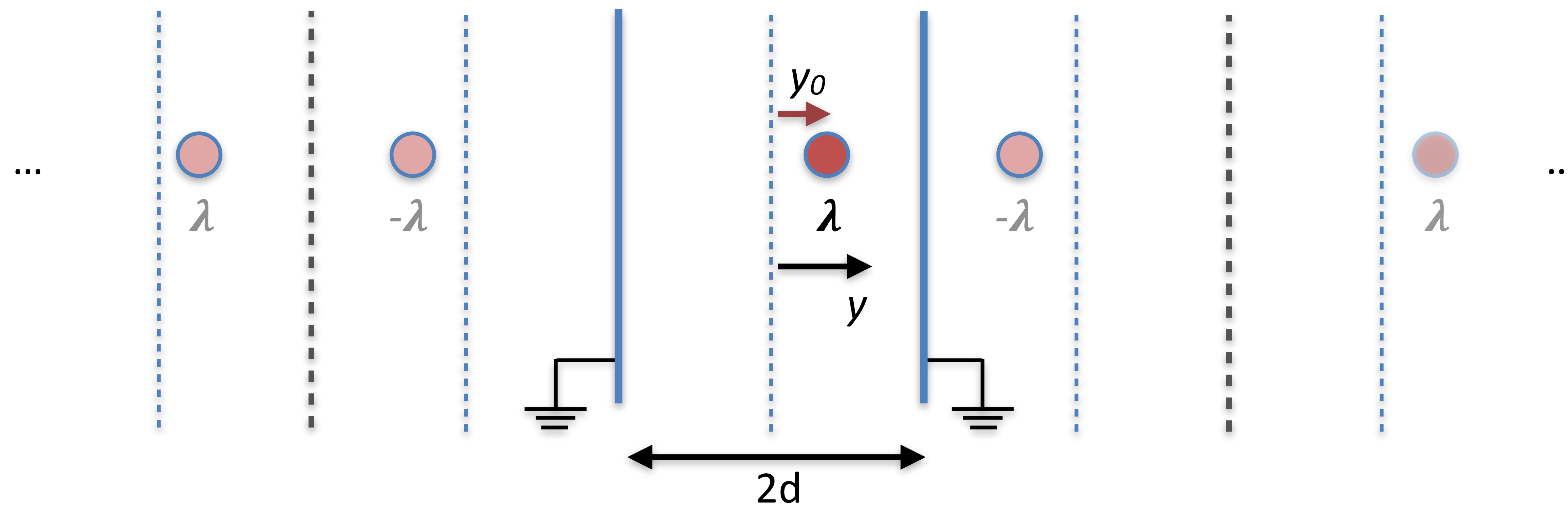


Image Charges and Image Currents

Actually,



from the images:

$$
\begin{aligned}
E &= \sum_i \frac{\lambda_i}{2\pi\epsilon_0 r_i} \\
&= \frac{\lambda}{2\pi\epsilon_0}\left[\frac{1}{(2d-y_0)-y} - \frac{1}{(2d+y_0)+y} + \frac{1}{(4d-y_0)-y} - \frac{1}{(4d+y_0)-y} + \dots\right] \\
&= \frac{\lambda}{2\pi\epsilon_0}\left[\frac{2(y_0+y)}{4d^2-(y_0+y)^2} + \frac{2(y_0-y)}{16d^2-(y_0-y)^2} + \frac{2(y_0+y)}{36d^2-(y_0+y)^2} + \dots\right] \\
&\approx \frac{\lambda}{4\pi\epsilon_0 d^2}\left[\frac{y_0+y}{1} + \frac{y_0-y}{4} + \frac{y_0+y}{9} + \frac{y_0-y}{16} + \dots\right] \\
&= \frac{\lambda}{4\pi\epsilon_0 d^2}\left[\left(1+\frac{1}{4}+\frac{1}{9}+\dots\right)y_0 + \left(1-\frac{1}{4}+\frac{1}{9}-\dots\right)y\right] \\
&= \frac{\lambda}{4\pi\epsilon_0 d^2}\left(\frac{\pi^2}{6}y_0 + \frac{\pi^2}{12}y\right) \\
&= \frac{\pi\lambda}{24\epsilon_0 d^2}(y_0 + \frac{1}{2}y).
\end{aligned}
$$

suppose y_0 is created from steering effects, and $y_\beta = y - y_0$ is due to betatron oscillation within the bunch:

$$E \approx \frac{\pi\lambda}{48\epsilon_0 d^2}(3y_0 + y_\beta)$$

Image Charges and Image Currents

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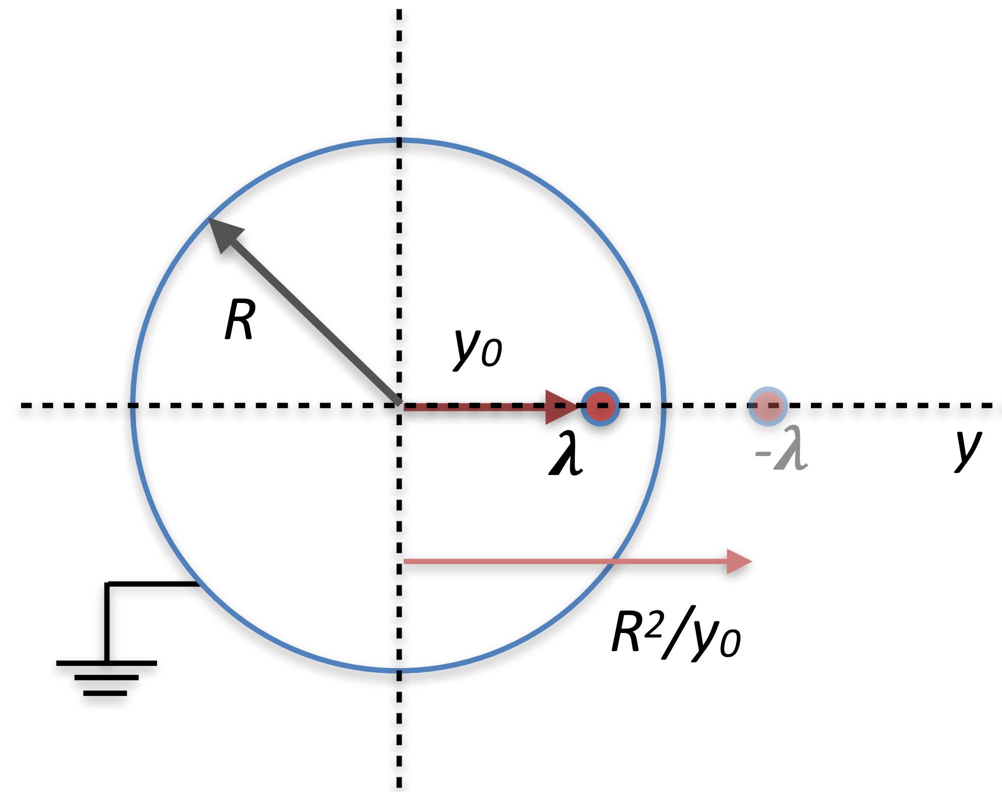
- Round Cylindrical Pipe:

gradient:

$$\frac{\partial E}{\partial y} = \frac{\lambda}{2\pi\epsilon_0}\frac{y_0^2}{R^4}$$

for small $y_0 \ll R$



- Such image charges and currents cause steering fields, tune shifts, etc., which vary with the beam current/intensity

Longitudinal Space Charge

- So far have made approximations that the charge density is independent of *s* (λ = constant)
- Wish to consider what happens when this symmetry is broken and we have disturbances such that
  - $\lambda = (d\lambda/ds)(s-s_0) = \lambda' \, (s-s_0)$
- The local space charge force on one side of the disturbance (s - $ds/2$) will be larger than the space charge force at $s + ds/2$ and the particles within the region *ds* will experience a net longitudinal force

Longitudinal Space Charge

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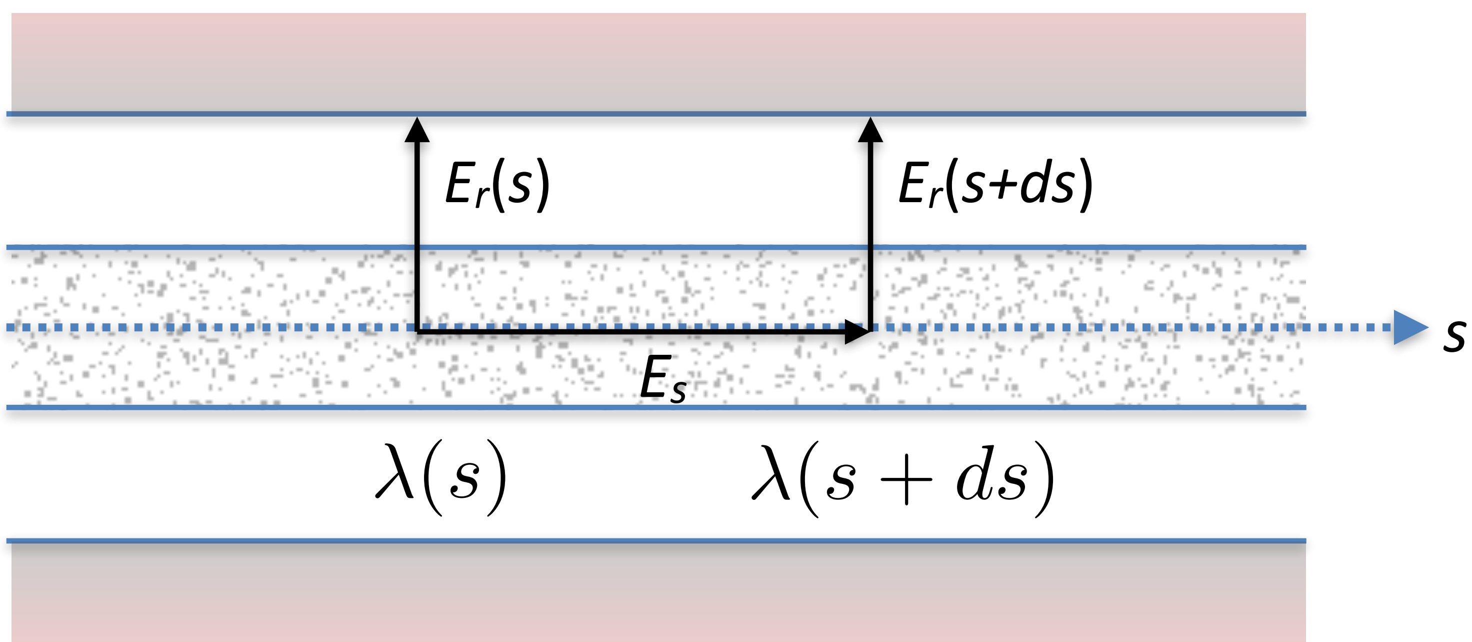

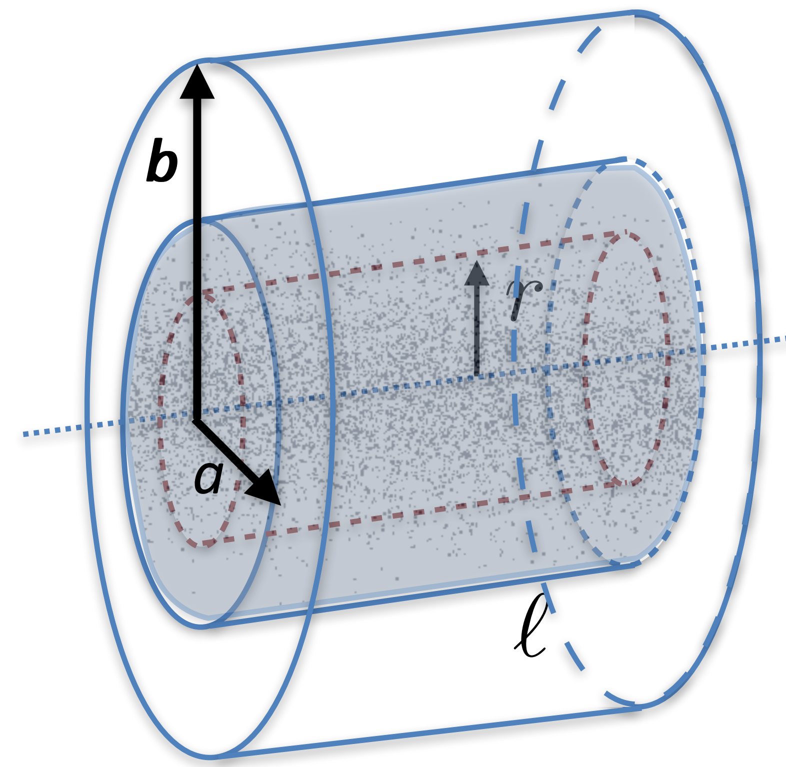

$$E_r = \frac{\lambda}{2\pi\epsilon_0}\,\frac{1}{r}, \qquad B_\phi = \frac{\mu_0 \lambda v}{2\pi}\,\frac{1}{r}, \qquad r \geq a;$$

$$E_r = \frac{\lambda}{2\pi\epsilon_0}\,\frac{r}{a^2}, \qquad B_\phi = \frac{\mu_0 \lambda v}{2\pi}\,\frac{r}{a^2}, \qquad r \leq a;$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{A}$$

Longitudinal Space Charge

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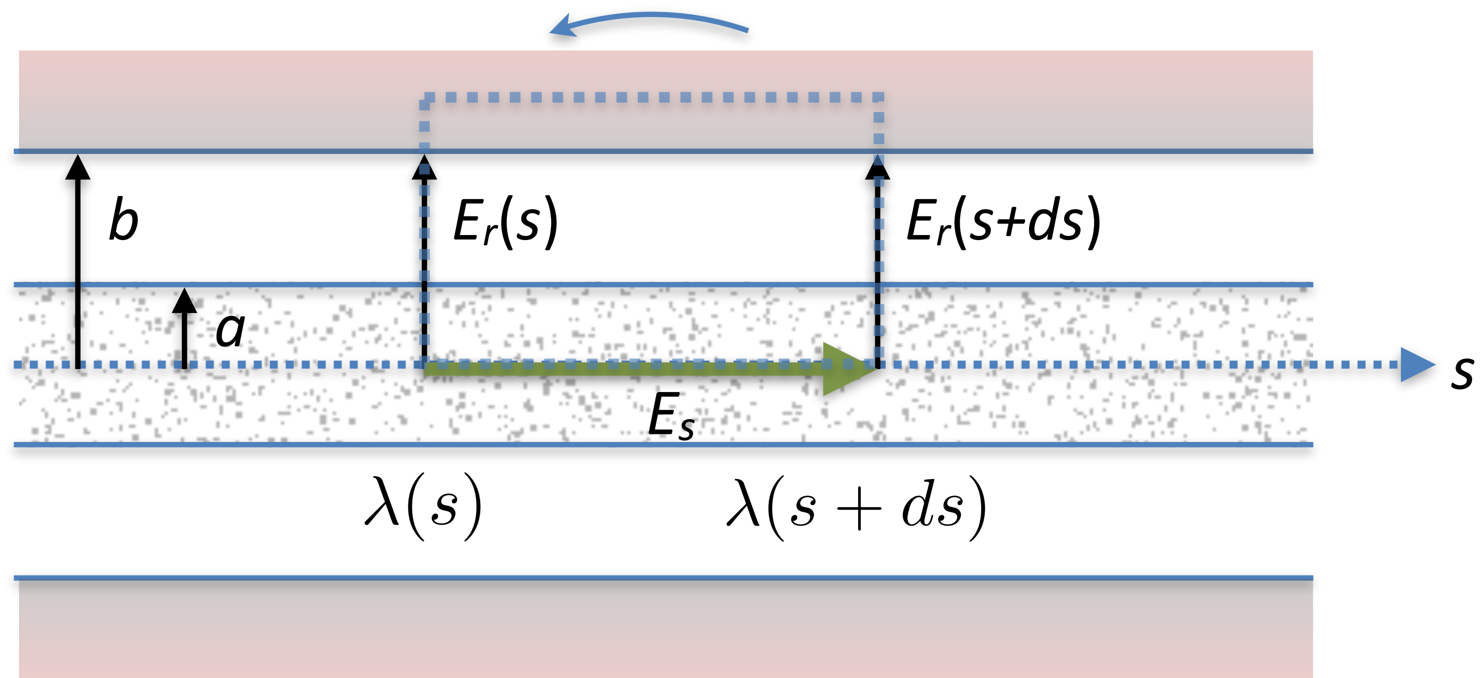
$$E_r = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}, \quad B_\phi = \frac{\mu_0 \lambda v}{2\pi}\frac{1}{r}, \quad r \geq a;$$

$$E_r = \frac{\lambda}{2\pi\epsilon_0}\frac{r}{a^2}, \quad B_\phi = \frac{\mu_0 \lambda v}{2\pi}\frac{r}{a^2}, \quad r \leq a;$$



$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A}$$

$$\lambda' = d\lambda/ds \qquad g_0 \equiv 1 + 2\ln\frac{b}{a}$$

LHS yields,

$$E_s ds + \frac{\lambda' ds}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)$$

RHS yields,

$$\left[\frac{\lambda' ds}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\left(\frac{v}{c}\right)^2\right]$$

$$E_s = -\frac{g_0 \lambda'}{4\pi\epsilon_0\gamma^2}$$

Longitudinal Space Charge

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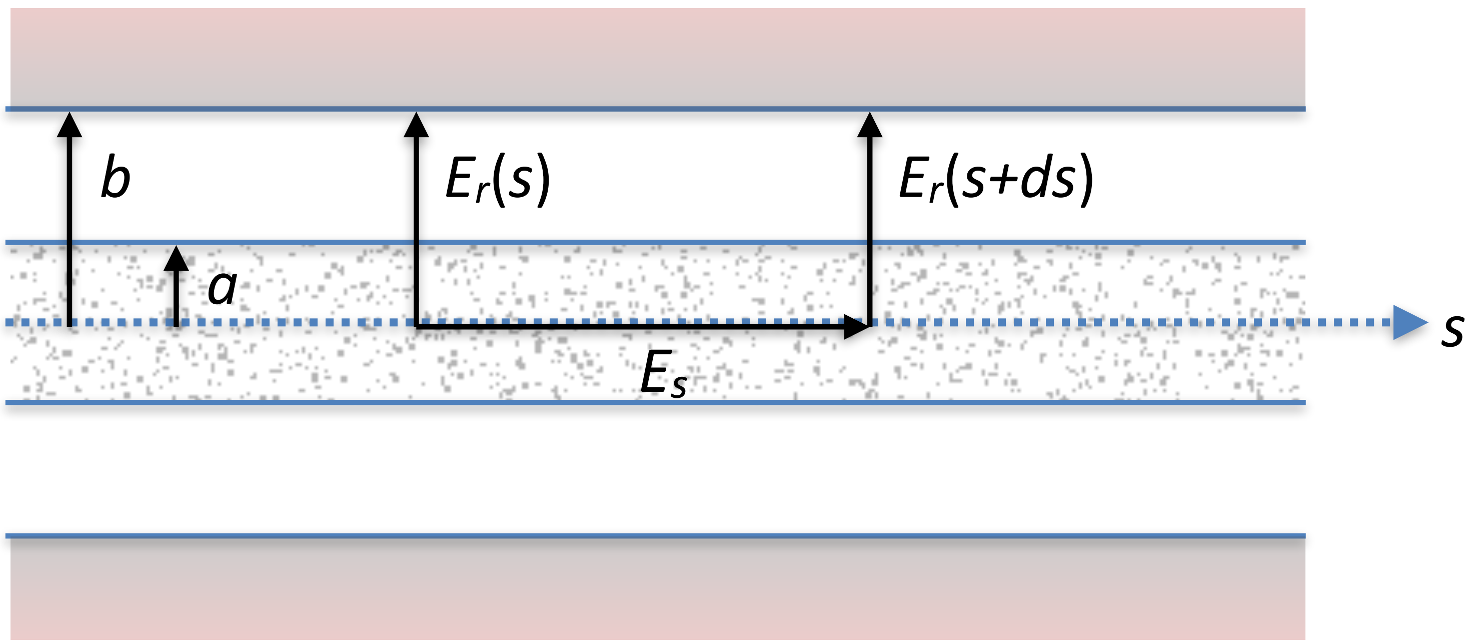
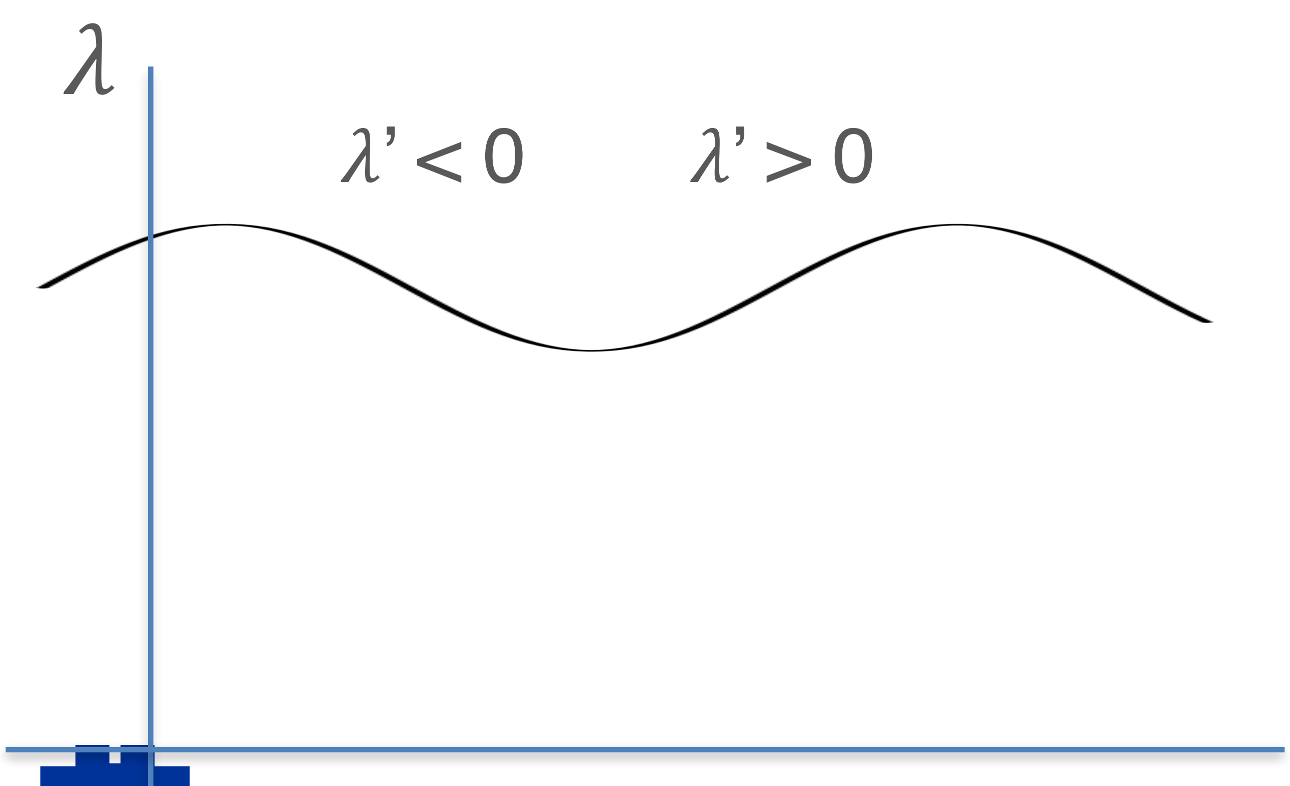


$$\lambda' = d\lambda/ds$$

$$g_0 \equiv 1 + 2\ln\frac{b}{a}$$

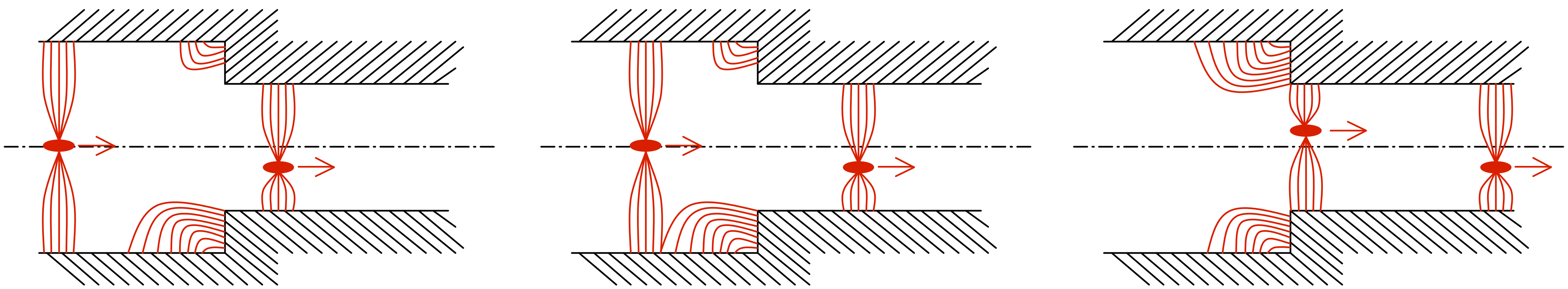
$$E_s = -\frac{g_0\lambda'}{4\pi\epsilon_0\gamma^2}$$



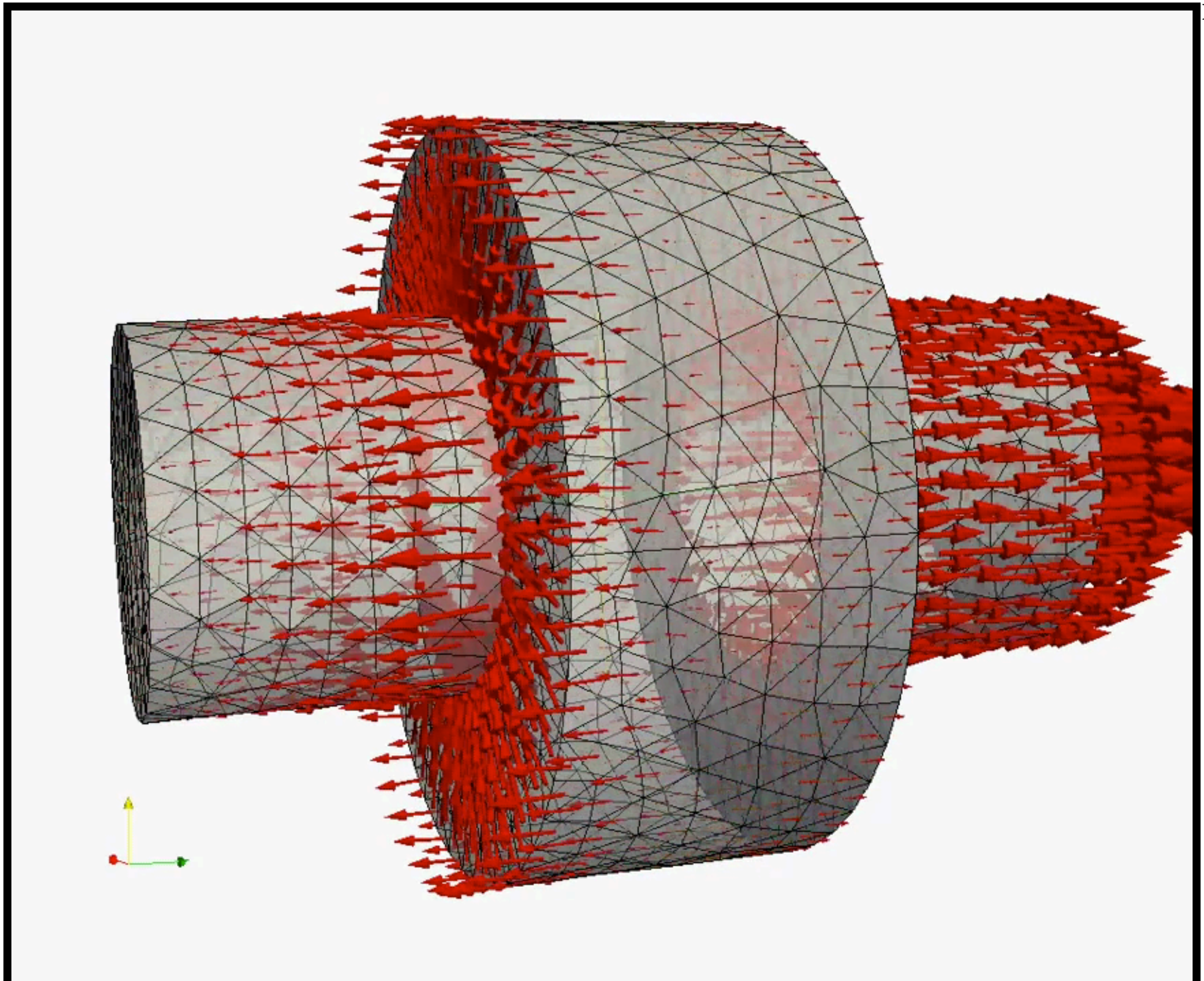
In a synchrotron, $E_s > 0$ will increase the speed of the particles. Below the transition energy, this increases their revolution frequencies and high density particles will move toward the "trough" of the local disturbance. Above transition, the revolution frequency will decrease, and perturbations can **grow**; this instability above transition is called the "negative mass" instability.

Wake Fields

- Wake fields are transient fields generated during the beam passage
  - Duration depends on the geometry & material of the structure
  - Case 1: Wake persists for the duration of a bunch passage
    - Particles in the tail can interact with wakes due to particles in the head.
    - *Single bunch instabilities* can be triggered
      - (distortion of the longitudinal distribution, bunch lengthening, transverse instabilities)



  - Case 2: The wake field lasts longer than the time between bunches
    - Trailing bunches can interact with wakes from leading bunches to generate ***multi-bunch*** *instabilities*



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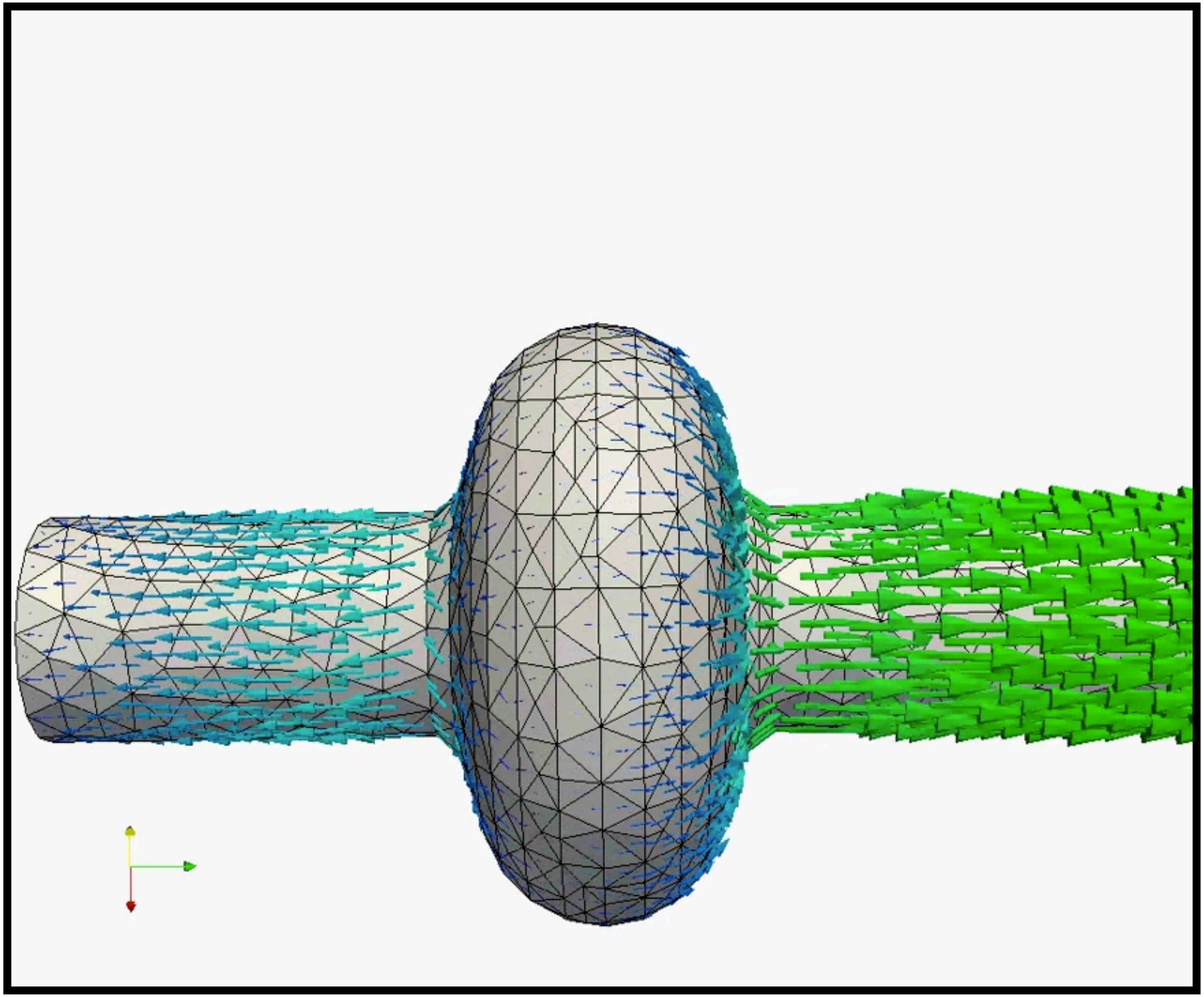

Wake Potentials

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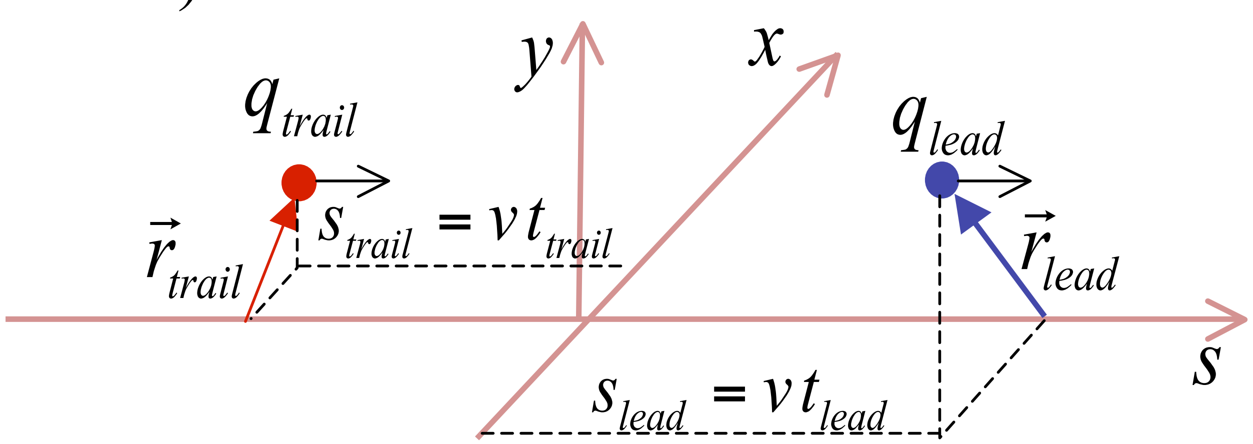
- Wake fields effects can be longitudinal or transverse.
  - → Longitudinal wakes change the energy of beam particles
    - For longitudinal wakes it suffices to consider *only its electric field*
  - → Transverse wakes affect beam particles' transverse momentum
- The wake potential is the energy variation induced by the wake field of the lead particle on a *unit charge* trailing particle

(Assume *v* constant.)



$$V_W\left(\vec{r}_{lead},\vec{r}_{trail},t_{trail}-t_{lead}\right) = \int_{-\infty}^{\infty} \vec{E}_W\left(s,\vec{r}_{lead},\vec{r}_{trail},t_{trail}-t_{lead}\right)\cdot d\vec{s}$$

Impedance

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- The wake function describes the interaction of the beam with its external environment in the *time domain*
- The frequency domain "alter ego" of W is the coupling impedance (in Ohms) and defined as the *Fourier transform of the wake function*

$$Z(\vec{r},\vec{r}_{trail},\omega)=\int_{-\infty}^{\infty} W(\vec{r},\vec{r}_{trail},\tau)e^{-j\omega\tau}\,d\tau \qquad with \quad \tau = t_{trail}-t$$

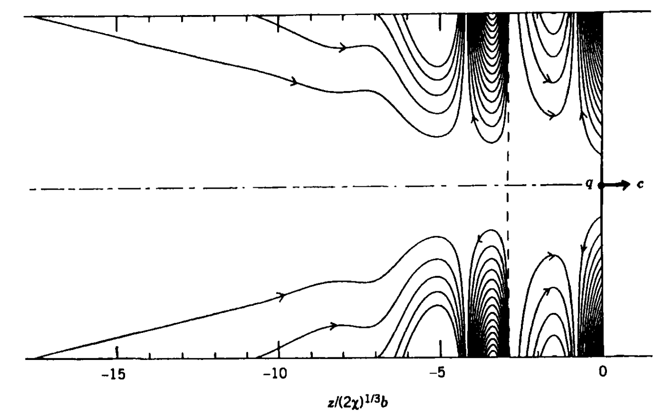
- If I is the Fourier transform of the charge distribution, the Fourier transform of the total induced voltage is simply given by:

$$\tilde{V}(\vec{r},\vec{r}_{trail},\omega)=Z(\vec{r},\vec{r}_{trail},\omega)\; I(\vec{r},\omega)$$

- Then

$$V(\vec{r},\vec{r}_{trail},\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{V}(\vec{r},\vec{r}_{trail},\omega)\,e^{j\omega\tau}\,d\omega$$

Transverse Wake Fields



- Transverse wake function is the transverse momentum kick per unit leading charge and unit trailing charge due to the wake fields

- Transverse wake fields are excited when the beam passes off center
  - For small displacements only the *dipole* term proportional to the displacement is important.
  - The *transverse dipole wake function* is the transverse wake function per unit displacement

- The transverse coupling impedance is defined as the Fourier transform of the transverse wake function times *j*

- Longitudinal and transverse wakes represent the same 3D wake field
  - Linked by Maxwell's equations.
  - The Panofsky-Wenzel relations allow one to calculate one wake component when the other is known.

Wake Fields in Real Accelerators

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- Accelerator vacuum chambers have complex shapes that include many components that can potentially host wake fields

- Not all wakes excited by the beam can be trapped in the chamber

- Given a chamber geometry, ∃ a cutoff frequency, f_{cutoff}
  - → Modes with frequency > f_{cutoff} propagate along the chamber

$$f_{Cutoff} \approx \frac{c}{b} \qquad where\ \ b \equiv transverse\ chamber\ size$$

Beam Break-Up

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- Transverse wake fields generated by head of the bunch, or by the beginning of a bunch train, will act on the particles at the tail of the bunch, or on the bunches at the tail-end of the train
-

Wake Force: $F_r = eQ_1 W_1(s)$

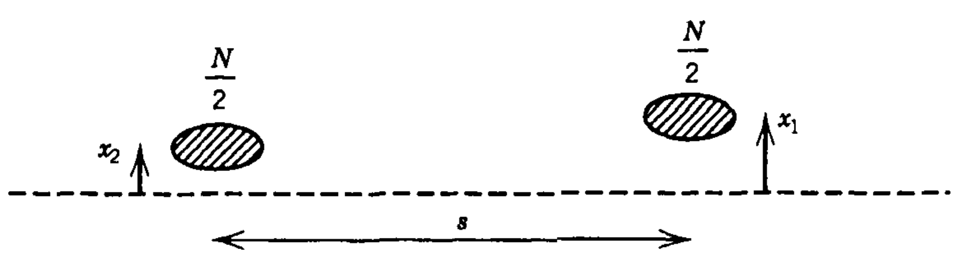


Figure 6.9. Two macroparticles, separated by a distance s, each containing half the particles of the bunch.

leading macroparticle is executing betatron oscillations according to

$$x_1 = \hat{x} \cos \omega_\beta t,$$

then the equation of motion of the second macroparticle becomes

$$\ddot{x}_2 + \omega_\beta^2 x_2 = \frac{Ne^2 W_1}{2m\gamma} x_1$$
$$= \frac{Ne^2 W_1}{2m\gamma} \hat{x} \cos \omega_\beta t.$$

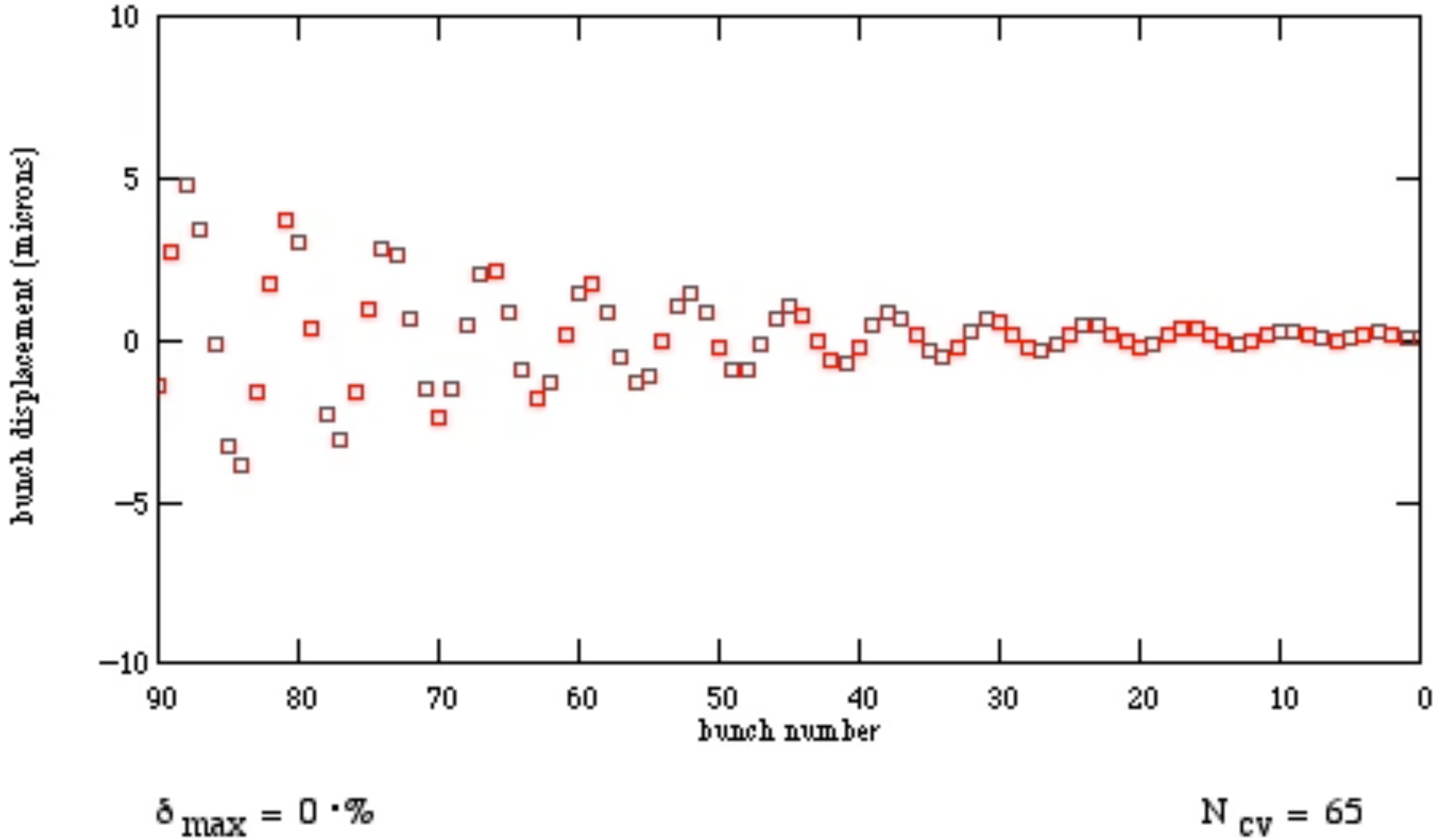
driven harmonic oscillator

Solution:

$$x_2(t) = \hat{x}_2 \cos \omega_\beta t + \hat{x}_1 \frac{Ne^2 W_1}{4\omega_\beta m\gamma} t \sin \omega_\beta t.$$

See Movie…

Beam Break-Up



Damping of Coherent Instabilities

- Instabilities of these sort generally caused by the source frequency (head of the bunch, say) having natural frequencies (betatron frequencies, say) similar to those of the trailing particles — thus, generate resonance conditions
- By creating a frequency spread across the distribution of particles, these resonant conditions often can be mitigated by
  - increasing momentum spread within a bunch, and/or adjusting chromaticity
  - generating a momentum spread across bunch train

Damping of Coherent Instabilities

Northern Illinois University

Let's go back to the driven harmonic oscillator:

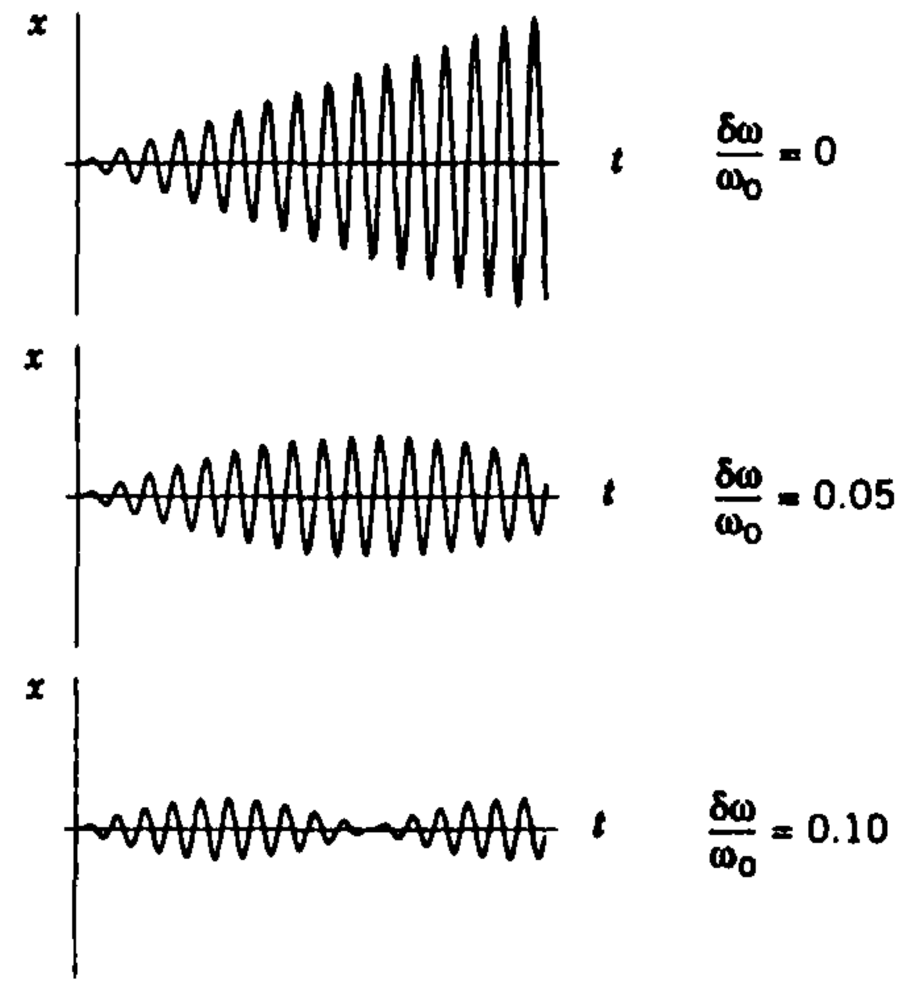
$$\ddot{x} + \omega_0^2 x = C \sin \omega t. \tag{6.195}$$

Consider first the situation on resonance, where $\omega = \omega_0$. For a particle starting from rest, the solution to the above differential equation is

$$x = \frac{C}{2\omega_0^2} \sin \omega_0 t - \frac{C}{2\omega_0} t \cos \omega_0 t. \tag{6.196}$$

Clearly, the envelope of the oscillation grows without bound. Off resonance,

$$\begin{aligned} x &= \frac{C}{\omega_0^2 - \omega^2}\left(\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t\right) \\ &= \frac{C}{(\omega_0 + \omega)\omega_0} \sin \omega_0 t \\ &\quad - \frac{C}{(\omega_0 + \omega)} t \left[\frac{\sin \frac{1}{2}\delta\omega\, t}{\frac{1}{2}\delta\omega\, t}\right] \cos\left(\frac{\omega + \omega_0}{2} t\right), \end{aligned} \tag{6.197}$$

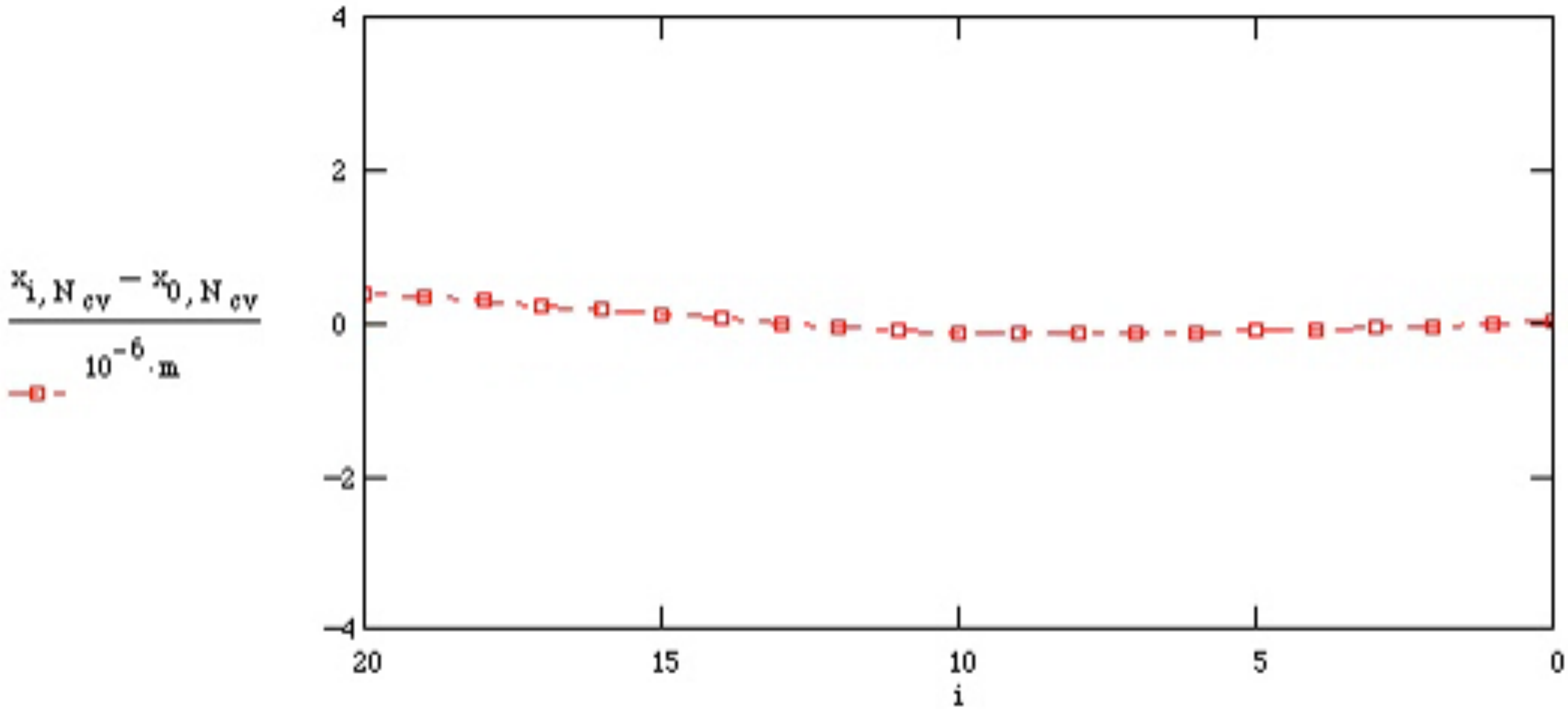


ex: transverse wake fields driven by average offsets of the particles

$$\ddot{x} + \omega^2 x = c\langle x \rangle$$

if increase frequency spread of particles, then will not grow coherently, and thus reduce <x> in the driving term

Mitigation with Energy Spread



$\delta_{max} = 10 \cdot \%$

$N_{cv} = 91$

Summary

- Particles interact electromagnetically with other nearby particles — space charge force
- Effects decrease as energy increases, typically as $1/\gamma^2$
- Manifestations:
  - defocusing effects; tune shifts and tune spreads
    - tune space (resonances) can be limited; limits intensity
  - beam-beam collisions — acts like an extra "lens"
    - does not vary with energy — only phase space density
  - possible instabilities
- Particles interact electromagnetically with their environment
  - images/fields in beam pipes can be strong — similar effects
- Intense beams create wake fields, interact back with particles
- Instabilities often mitigated by creating higher tune spread within the the beam distribution (momentum spread, chromaticity