Assignment: HW2 [50 points]

Assigned: 2016/09/13
Due: 2016/09/22

P2.1 [$1 + 3 + 3 + 3 + 2 = 12$ points]
Let the motion of a point mass be governed by the law

$$\ddot{\mathbf{r}} = \dot{\mathbf{r}} \times \mathbf{a}, \qquad \mathbf{a} = \text{ const.} \tag{1}$$

(a) Show that $\dot{\mathbf{r}} \cdot \mathbf{a}$ is constant in time.

(b) Reduce Eq. 1 to an inhomogeneous differential equation of the form $\ddot{\mathbf{r}} + \omega^2 \mathbf{r} = \mathbf{f}(t)$.

(c) Solve the above equation by using a particular function of the form $\mathbf{r}_p = \mathbf{c}t + \mathbf{d}$.

(d) Express the constants of integration in terms of the initial values $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$.

(e) Describe the curve $\mathbf{r}(t) = \mathbf{r}_g(t) + \mathbf{r}_p(t)$, where $\mathbf{r}_g(t)$ is the solution of the homogeneous equation $\ddot{\mathbf{r}} + \omega^2 \mathbf{r} = 0$.

P2.2 [$2 + 5 = 7$ points]
Use variational calculus to find

(a) The shortest connection between two points (x_1, y_1) and (x_2, y_2) on the 2-dimensional Euclidean plane.

(b) The shape of a line of uniform mass density supported between (x_1, y_1) and (x_2, y_2) in a uniform gravitational field $\mathbf{g} = g\hat{\mathbf{e}}_y$.

P2.3 [$2 + 3 = 5$ points]
If a conservative force field in 3-dimensional space is axially symmetric, and we choose the z axis of a cylindrical coordinate system $\{r, \phi, z\}$ along the axis of symmetry, then show that

(a) The corresponding potential has the form $U = U(r, z)$,

(b) The force always lies in a plane containing the z axis.

P2.4 [$3 + 4 + 3 + 5 = 15$ points]
A bead of mass m slides without friction in a uniform gravitational field of acceleration $\mathbf{g}$ on a vertical circular hoop of radius R. The hoop is constrained to rotate at a fixed angular velocity Ω about its vertical diameter. Take the center of the hoop as the pole (origin) of a spherical polar coordinate system in which $\mathbf{r} = \{r, \theta, \phi\}$ represents the radius vector of the bead, with $\theta = 0$ along the direction of gravity.

(a) Write down the Lagrangian $L(\theta, \dot{\theta})$.

(b) Find how the equilibrium values of θ depend on Ω. Which are stable and which are unstable?

(c) Find the frequencies of small vibrations about the stable equilibrium positions (hint: use the first term in a Taylor series expansion). What happens when $\Omega = \sqrt{\dfrac{g}{R}}$?

(d) Is

$$H \equiv \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \tag{2}$$

conserved? Evaluate H, compare it to the energy $E \equiv T + V$, and comment on the result.

P2.5 $[3 + 6 + 2 = 11$ points$]$
The Lagrangian for a particle of mass m and electric charge e moving in the xy plane is given by

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + eEy - \frac{eB}{c}y\dot{x}.$$

This describes the motion of the particle in a uniform electric field E in the y direction and a uniform magnetic field B in the z direction. c is the speed of light.

(a) Write down the Euler-Lagrange equations.

(b) Find the Hamiltonian. Simplify the expression (eliminating $\dot{x}$ by making use of a first integral of motion). What can you say about the general allowed motions in $y(t)$?

(c) At $t = 0$, $x = y = 0$. What critical value v_c must $\dot{x}(0)$ take in order for the particle to go in a uniform motion? What is the corresponding value of $\dot{y}$? (You can either make use of the results in part (a) or that in part (b).)